# Fractional Cauchy problems on bounded domains: survey of recent results

Erkan Nane

DEPARTMENT OF MATHEMATICS AND STATISTICS
AUBURN UNIVERSITY

July 28-31, 2010

# Probability and transforms

If the random variable $X$ has density $f(x)$ so that

$$P(a \le X \le b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\begin{aligned}\hat{f}(k) = E(e^{-ikX}) &= \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \\ &= \int_{-\infty}^{\infty} (1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots)f(x)dx \\ &= 1 - ik\mu_1 - \frac{1}{2!}k^2\mu_2 + \cdots\end{aligned}$$

where the $l$th moment is $\mu_l = \int_{-\infty}^{\infty} x^l f(x)dx$

# Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\hat{f}(k) = 1 - k^2 + \cdots$
The IID sum $S(n) = X_1 + \cdots + X_n$ has FT $\hat{f}(k)^n$ and the normalized sum $S(n)/\sqrt{n}$ has FT

$$\left(\hat{f}(k/\sqrt{n})\right)^n = (1 - (k/\sqrt{n})^2 + \cdots)^n$$
$$= \left(1 - \frac{k^2}{n} + \cdots\right)^n$$
$$\to e^{-k^2} \equiv \hat{g}(k) \quad \text{as} \quad n \to \infty.$$

Inverting the Fourier transform reveals a Gaussian(Normal) density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$

# Brownian motion

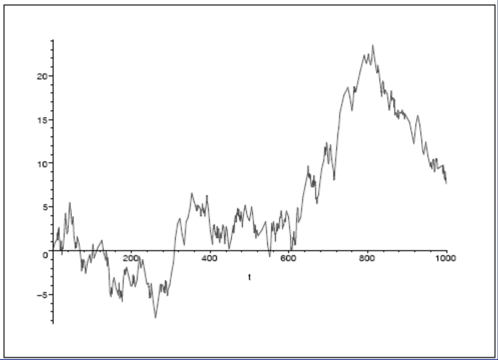
$X_n$ particle jump at time $n$ then
$S(n) = X_1 + \cdots + X_n$ is the location of the particle at time $n$.
Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2} S([ct]) \implies B(t)$ since

$$\hat{f}(c^{-1/2}k)^{[ct]} = \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]}$$
$$\to e^{-k^2 t} \equiv \hat{p}(t,k) \quad \text{as} \quad c \to \infty$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process $B(t)$ is Gaussian (Normal)

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

# Scaling limit: Brownian motion




Random graph of fractal dimension 1.5 and no jumps.


Most likely shape of a Brownian path.

Microsoft stock-the last two years

# Derivatives and transforms

- If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t)dt$$

then $\frac{df(t)}{dt}$ has Laplace transform $s\tilde{f}(s) - f(0)$.

- If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x)dx$$

then $\frac{df(x)}{dx}$ has Fourier transform $ik\hat{f}(k)$.

# The diffusion (heat) equation

Taking Fourier transforms in the classical diffusion equation

$$\partial_t p(t,x) = \partial_x^2 p(t,x)$$

yields

$$\partial_t \hat{p}(t,k) = (ik)^2 \hat{p}(t,k) = -k^2 \hat{p}(t,k)$$

whose solution

$$\hat{p}(t,k) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion $B(t)$.
For a cloud of diffusing particles $p(t,x)$ is the particle density.

# Brownian motion and diffusion (heat) equation

Let $B(t) \in \mathbb{R}$ be Brownian motion started at $x$. Then the function (convolution of f and $p(t,x)$)

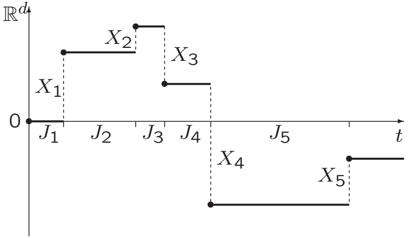
$$u(t,x) = \mathbb{E}_x[f(B(t))] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)dy$$

solves the heat equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x), \qquad t>0, \quad x \in \mathbb{R} \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned}$$

This is due to J.L. Doob (1956).
In this case we say, Brownian motion $B(t)$ is a stochastic solution of the heat equation.

## Continuous time random walks




The CTRW is a random walk with jumps $X_n$ separated by random waiting times $J_n$. The random vectors $(X_n, J_n)$ are i.i.d.

# Waiting time process

$J_n$'s are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the $n$th jump.
$N(t) = \max\{n \geq 0 : \ T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a $\beta$-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \quad D(u) > t\}.$$

The self-similar limit $E(ct) \stackrel{d}{=} c^{\beta} E(t)$ is non-Markovian.

# Continuous time random walks (CTRW)

Particle jump random walk has scaling limit
$c^{-1/2}S([ct]) \implies B(t)$.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies E(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

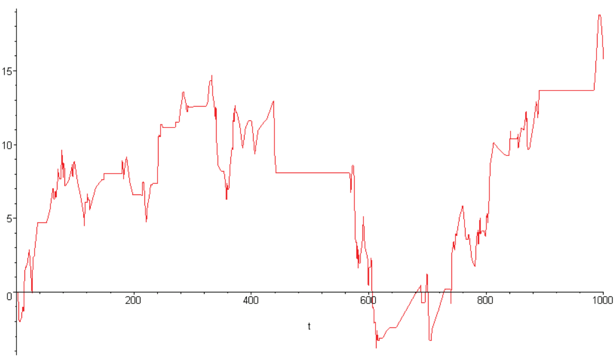
$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}$$

CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta}\cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \overset{d}{=} c^{\beta/2}B(E(t))$ is non-Markovian.

# Scaling limit: Subordinated motion




Limit retains long waiting times.

# Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

# Fractional derivatives: An old idea gets new life

- Fractional derivatives $D^\beta f(x)$ for any $\beta > 0$ were invented by Leibniz (1695) soon after the more familiar integer derivatives.
- The Caputo fractional derivative of order $0 < \beta < 1$ defined by

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta} \qquad (1)$$

  was invented to properly handle initial values (Caputo 1967).
- Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

# examples

- $$D_t^{\beta}(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^{\beta}(e^{\lambda t}) = \lambda^{\beta} e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^{\beta}(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

# Time-fractional model for anomalous sub-diffusion

Nigmatullin (1986), Zaslavsky (1994) studied the Cauchy problem

$$\partial_t^\beta u(t,x) = \partial_x^2 u(t,x); \quad u(0,x) = f(x) \tag{2}$$

that models particles that wait for a time $J_n$ before the $n$th jump, where $P(J_n > t) \approx t^{-\beta}$ for some $0 < \beta < 1$.
The solution to time-fractional diffusion is given by

$$u(t,x) = \mathbb{E}_x(f(B(E(t)))) = \int_0^\infty p(l,x) g_{E(t)}(l) dl$$

$p(l,x) = \mathbb{E}_x(f(B(l)))$ solution to the heat equation.
$g_{E(t)}(l)$ density of $E(t)$
In finance, $E(t)$ represents the number of trades by time $t$.
$\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx xt^{\beta/2}$.

# Heat equation in bounded domains

Heat equation in $D$ with Dirichlet boundary conditions:

$$\begin{aligned}\partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D,\ t > 0,\\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D.\end{aligned}$$

When $D = (0, M)$ can be solved by separation of variables: set $u(t,x) = \phi(x)T(t)$. Hence $\phi(x)$ satisfies

$$\partial_x^2 \phi(x) = -\lambda \phi(x), \quad x \in (0, M), \lambda > 0; \quad \phi(0) = 0, \quad \phi(M) = 0$$

and $T(t)$ satisfies $T'(t) = -\lambda T(t); \quad T(0) = 1.$

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}, T_n(t) = e^{-\lambda_n t}.$$

Same applies in any dimension $d \geq 1$.

# Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of $\Delta$ on a bounded domain $D$ with Dirichlet boundary conditions by $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta\phi_n(x) = -\lambda_n\phi_n(x), \quad x \in D; \phi_n(x) = 0, \quad x \in \partial D.$$

$\tau_D(X) = \inf\{t \geq 0 : \ X(t) \notin D\}$ is the first exit time of a process $X$, and let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \;=\; \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in $D$ with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

# Fractional diffusion in bounded domains

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{3}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \\ &= \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \end{aligned}$$

Joint work with Meerschaert and Vellaisamy, AOP (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).
In this case, eigenfunctions and eigenvalues are

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}$$

The time fractional diffusion on $(0, M)$ has the solution

$$u(t,x) \;=\; \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) E_\beta(-\lambda_n t^\beta)$$

here

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}$$

For $\beta = 1$, $E_1(-z) = e^{-z}$, and $u$ coincides with the solution of the heat equation on $(0, M)$.

# Sketch of Proof

- Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta_x u(t,x)dx = \int_D u(t,x)\Delta_x \phi_n(x)$$
$$= -\lambda_n \int_D u(t,x)\phi_n(x)dx = -\lambda_n \bar{u}(t,n)$$

Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\lambda_n \bar{u}(t,n). \tag{4}$$

- taking Laplace transforms on both sides of (4), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\lambda_n \hat{u}(s,n) \tag{5}$$

- Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\lambda_n}$.

# Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n)E_\beta(-\lambda_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\lambda E(t)}) = \int_0^\infty e^{-\lambda l} g_{E(t)}(l)dl = E_\beta(-\lambda t^\beta).$$

Inverting now the $\phi_n$-transform, we get an $L^2$-convergent solution of Equation (3) as (for each $t \geq 0$ )

$$u(t,x) = \sum^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \quad (6)$$

# Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) \int_0^\infty e^{-\lambda_n l} g_{E(t)}(l)dl \\
&= \int_0^\infty \left( \sum_{n=1}^{\infty} \bar{f}(n)e^{-\lambda_n l}\phi_n(x) \right) g_{E(t)}(l)dl \qquad (7) \\
&= \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\
&= \int_0^\infty \mathbb{E}_x[f(B(l))I(\tau_D > l)]g_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))]
\end{aligned}
$$

# Stochastic model for ultraslow diffusion

Let $supp\mu \subset (0,1)$ be a finite measure. $B_i$ iid with dist. $\mu$.
$J_i^c \overset{d}{=} c^{-1/\beta} J_1$ nonnegative iid with
$P(J_1 > t) = \int_0^1 t^{-\beta}\mu(d\beta), \quad t \geq 1$ and

$$P(c^{-1/\beta}J_1 > u | B_1 = \beta) = \begin{cases} 1, & 0 \leq u < c^{-1/\beta} \\ c^{-1}u^{-\beta}, & u \geq c^{-1/\beta} \end{cases}$$

Time of the $n$th jump at scale $c$ has
$T^{(ct)}(t) = \sum_{i=1}^{[ct]} J_i^c \implies W(t)$, increasing Lévy process.
The number of jumps by time $t \geq 0$ at scale $c$ has scaling limit;
$N_t^c = \max\{n \geq 0 : \ T^c(n) \leq t\} \implies E(t) = \inf\{\tau \geq 0 : \ W(\tau) \geq t\}$.
$X_i^c$ iid jumps has scaling limit
$S^c(ct) = X_1^c + X_2^c + \cdots + X_{[ct]}^c \implies B(t)$ then

$$S^c(N_t^c) \implies B(E(t))$$

# Ultraslow diffusion

$\mathbb{E}[e^{-sW_t}] = e^{-t\psi_W(s)}$ and Laplace exponent

$$\psi_W(s) = \int_0^\infty (e^{-sx} - 1)\phi_W(dx) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta). \quad (8)$$

The Lévy measure is

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\mu(d\beta), \quad (9)$$

Then stochastic model for **ultraslow** diffusion $B(E(t))$ is a stochastic solution of

$$\mathbb{D}^{(\nu)}u(t,x) := \int_0^1 \partial_t^\beta u(t,x)\nu(d\beta) = \Delta_x u(t,x); \nu(d\beta) = \Gamma(1-\beta)\mu(d\beta).$$

For special $\nu$: $\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx x(\log t)^{\beta/2}$.

# Ultraslow diffusion in bounded domains

$$\begin{aligned} \mathbb{D}^{(\nu)}u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{10}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h(t,\lambda_n) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\ &= \mathbb{E}_x[f(X(E(t)))I(\tau_D(X) > E(t))] \\ &= \mathbb{E}_x[f(X(E(t)))I(\tau_D(X(E)) > t)] \end{aligned}$$

where $h(t,\lambda_n) = \mathbb{E}(e^{-\lambda_n E(t)})$.

Joint work with Meerschaert and Vellaisamy (2009).

# Eigenvalue problem for distributed order time derivative

$h(t,\lambda) = \mathbb{E}(e^{-\lambda E(t)})$ is the solution of

$$\mathbb{D}^{(\nu)} h(t,\lambda) = -\lambda h(t,\lambda); \quad h(0,\lambda) = 1. \tag{11}$$

In the case $\mu(d\beta) = p(\beta)d\beta$; by inverse Laplace transform it has the representation

$$h(t,\lambda) = \frac{\lambda}{\pi}\int_0^\infty r^{-1} e^{-tr}\Phi(r,1)dr \tag{12}$$

where for $U(r) = \int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta$

$$\Phi(r,1) = \frac{U(r)}{[\int_0^1 r^\beta \cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \lambda]^2 + [U(r)]^2}.$$

Due to Kochubei (2008).

# Equivalence to Higher order PDE's

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^m u(t,x); \quad u(0,x) = f(x) \tag{13}$$

and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{14}$$

have the same unique solution given by

$$u(t,x) = \int_0^\infty p(s,x) g_{E(t)}(s)\, ds = \mathbb{E}_x(f(B(E(t))))$$

- Due to Allouba and Zheng (2001), Baeumer, Meerschaert, and Nane (2007), Keyantuo and Lizama (2009).

# Connections to iterated Brownian motions

- Orsingher and Benghin (2004) and (2008) show that the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{15}$$

is given by running

$$I_{n+1}(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where $B_j$'s are independent Brownian motions, i.e., $u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (15), and solves (13) for $m = 2^n$.

# IBM in bounded domains

The (classical) solution of

$$\begin{aligned} \partial_t u(t,x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^{2^n} u(t,x), \quad x \in D, \ t > 0; \\ u(t,x) &= \Delta_x{}^l u(t,x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1; \\ u(0,x) &= f(x), \ x \in D \end{aligned}$$

is given by (running

$I_{n+1}(t) = B_1(|B_2(\cdots |B_{n+1}(t)|)|) = B_1(|I_n(t)|))$

$$\begin{aligned} u(t,x) &= \mathbb{E}_x[f(I_{n+1}(t)) I(\tau_D(B_1) > |I_n(t)|)] \\ &= 2\int_0^\infty T_D(l) f(x) h(t,l) dl, \end{aligned} \tag{16}$$

where $h(t,l)$ is the transition density of $\{I_n(t)\}$.

Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$.

# Extensions: Markov generators

The Markov process $X(t)$ with generator

$$L_x u = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j} u + \sum_{i=1}^{d} b_i(x)\partial_{x_i} u \tag{17}$$

solves $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt$ with $a = \sigma\sigma^T$. Then

$$p(t,x) = T_D(t)f(x) = \mathbb{E}_x[f(X(t))I(\tau_D(X) > t)]$$

solves the Cauchy problem

$$\partial_t p(t,x) = L_x p(t,x)$$

with Dirichlet boundary conditions.

# Further research

- Extension to Neumann boundary conditions...
- Extension to $\beta > 1$
- Extensions to other time operators; tempered fractional derivative...

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} g(t) = e^{-\lambda t}\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{e^{\lambda s} g(s)\,ds}{(t-s)^\beta} - \lambda^\beta g(t) - \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr.$$

- Fractal properties of $B(E(t))$ and other subordinate processes
- Work in progress for the Subordinated Brownian motions, e.g. symmetric stable process as the outer process....
- Applications-interdisciplinary research

# Thank You!