

FRACTIONAL CAUCHY PROBLEMS

Erkan Nane

DEPARTMENT OF MATHEMATICS AND STATISTICS
AUBURN UNIVERSITY

September 3, 2008

Fractional differentiation

Fractional Diffusion

Initial-Boundary value problems

Two approaches

- ▶ Riemann-Liouville fractional derivative;

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^\beta} \right]$$

with laplace transform $s^\beta \tilde{g}(s)$. $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ denotes the usual Laplace transform of g .

- ▶ The Caputo fractional derivative of order $0 < \beta < 1$ defined by

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta} \quad (1)$$

was invented to properly handle initial values (Caputo 1967).

- ▶ Laplace transform of $\partial_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples



$$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$



$$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$



$$D_t^\beta(\sin t) = \sin\left(t + \frac{\pi\beta}{2}\right)$$

Cauchy Problems

Nigmatullin (1986) gave a Physical derivation of fractional diffusion

$$\frac{\partial^\beta}{\partial t^\beta} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x) \quad (2)$$

where $0 < \beta < 1$ and L_x is the generator of some continuous Markov process $X_0(t)$ started at $x = 0$.

Zaslavsky (1994) used this to model Hamiltonian chaos.

Stochastic solution

- ▶ Baeumer and Meerschaert (2001) and Meerschaert and Scheffler (2004) shows that, in the case $p(t, x) = T(t)f(x)$ is a bounded continuous semigroup on a Banach space (with corresponding process X_t , $E_t = \inf\{u : D_u > t\}$, D_t is a stable subordinator with index β), the formula

$$u(t, x) = E_x(f(X_{E_t})) = \frac{t}{\beta} \int_0^\infty p(s, x) g_\beta\left(\frac{t}{s^{1/\beta}}\right) s^{-1/\beta-1} ds$$

yields a solution to the **fractional Cauchy problem**:

$$\frac{\partial^\beta}{\partial t^\beta} u(t, x) = L_x u(t, x); \quad u(0, x) = f(x) \quad (3)$$

Here $g_\beta(t)$ is the smooth density of the stable subordinator, such that the Laplace transform

$$\tilde{g}_\beta(s) = \int_0^\infty e^{-st} g_\beta(t) dt = e^{-s^\beta}.$$

Proof

- ▶ We will use the following notation for the Laplace, Fourier, and Fourier-Laplace transforms (respectively):

$$\tilde{u}(s, x) = \int_0^{\infty} e^{-st} u(t, x) dt;$$

$$\hat{u}(t, k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} u(t, x) dx;$$

$$\bar{u}(s, k) = \int_{\mathbb{R}^d} e^{-ik \cdot x} \int_0^{\infty} e^{-st} u(t, x) dt dx.$$

Let ψ be the characteristic exponent of X_t .

- ▶ Taking Fourier transforms on both sides of (3) gives

$$\frac{\partial^\beta \hat{u}(t, k)}{\partial t^\beta} = \psi(k) \hat{u}(t, k)$$

Take Laplace transforms on both sides, using the fact that $s^\beta \tilde{g}(s) - s^{\beta-1}g(0)$ is the Laplace transform of the Caputo fractional derivative $\partial^\beta g(t)/\partial t^\beta$, to get

$$s^\beta \bar{u}(s, k) - s^{\beta-1} \hat{f}(k) = \psi(k) \bar{u}(s, k)$$

- ▶ and collect terms to obtain

$$\begin{aligned} \bar{u}(s, k) &= \frac{s^{\beta-1} \hat{f}(k)}{s^\beta - \psi(k)} \\ &= s^{\beta-1} \int_0^\infty \exp(-[s^\beta - \psi(k)]u) \hat{f}(k) du \end{aligned} \tag{4}$$

- ▶ $p(k, t) = \exp(\psi(k)t)\hat{f}(k)$ is the Fourier transform of $p(x, t)$
- ▶ Use $d(e^{-s^\beta u})/ds = -\beta s^{\beta-1} u e^{-s^\beta u}$ to get
- ▶ $s^{\beta-1} e^{-s^\beta u} = -(\beta u)^{-1} d(e^{-s^\beta u})/ds$
- ▶ recall $e^{-s^\beta} = \tilde{g}_\beta(s)$, Laplace transform of the density of strictly stable process of index β .
- ▶ $e^{-s^\beta u} = e^{-(su^{1/\beta})^\beta} = \int_0^\infty e^{-su^{1/\beta} v} g_\beta(v) dv = \int_0^\infty e^{-st} g_\beta(u^{-1/\beta} t) u^{-1/\beta} dt$
- ▶

$$\begin{aligned}
 s^{\beta-1} e^{-s^\beta u} &= -(\beta u)^{-1} d(e^{-s^\beta u})/ds \\
 &= (-1/\beta u) \frac{d}{ds} \left(\int_0^\infty e^{-st} g_\beta(u^{-1/\beta} t) u^{-1/\beta} dt \right) \\
 &= (1/\beta u) \int_0^\infty t e^{-st} g_\beta(u^{-1/\beta} t) u^{-1/\beta} dt
 \end{aligned}$$

- ▶ we obtain

$$\bar{u}(s, k) = \int_0^\infty e^{-st} \left(\int_0^\infty p(k, u) \frac{t}{\beta} g_\beta(u^{-1/\beta t}) u^{-1/\beta-1} du \right) dt$$

- ▶ Invert the Laplace transform to obtain

$$\hat{u}(t, k) = \int_0^\infty p(k, u) \frac{t}{\beta} g_\beta(u^{-1/\beta t}) u^{-1/\beta-1} du$$

- ▶ Then invert the Fourier transform to get

$$u(t, x) = E_x(f(X_{E_t})) = \frac{t}{\beta} \int_0^\infty p(s, x) g_\beta\left(\frac{t}{s^{1/\beta}}\right) s^{-1/\beta-1} ds$$

- ▶ This proof is due to Meerschaert, Benson, Scheffler and Baeumer (2002)
- ▶ Proof in the case $p(t, x)$ is the solution to an abstract Cauchy problem is by Baeumer and Meerschaert (2001)

Equivalence to Higher order PDE's

- ▶ For any $m = 2, 3, 4, \dots$ both the Cauchy problem

$$\frac{\partial u(t, x)}{\partial t} = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t, x); \quad u(0, x) = f(x) \quad (5)$$

and the fractional Cauchy problem:

$$\frac{\partial^{1/m} u(t, x)}{\partial t^{1/m}} = L_x u(t, x); \quad u(0, x) = f(x), \quad (6)$$

have the same unique solution given by

$$u(t, x) = \int_0^\infty p((t/s)^{1/m}, x) g_{1/m}(s) ds$$

- ▶ Due to Baeumer, Meerschaert, and Nane (2007).

Connections to iterated Brownian motions

- ▶ Orsingher and Benghin (2004) and (2008) show that for $\beta = 1/2^n$ the solution to

$$\frac{\partial^{1/2^n}}{\partial t^{1/2^n}} u(t, x) = \Delta_x u(t, x); \quad u(0, x) = f(x), \quad (7)$$

is given by running

$$I_n(t) = B_1(|B_2(|B_3(|\cdots (B_{n+1}(t)) \cdots |)|)|)$$

Where B_j 's are independent Brownian motions, i.e., $u(t, x) = E_x(f(I_n(t)))$ solves (7), and solves (5) for $m = 2^n$.

Corollary

- ▶ We obtain the equivalence of one dimensional distributions in the case E_t is the inverse stable subordinator of index $\beta = 1/2^n$

$$I_n(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|) \stackrel{(d)}{=} B_1(E_t)$$

Heat equation in bounded domains

Given a complete orthonormal basis $\{\psi_n(x)\}$ on $L^2(D)$, we will call

$$\bar{u}(t, n) = \int_D \psi_n(x) u(t, x) dx;$$

$$\hat{u}(s, n) = \int_D \psi_n(x) \int_0^\infty e^{-st} u(t, x) dt dx = \int_D \psi_n(x) \tilde{u}(s, x) dx.$$

the ψ_n , and ψ_n -Laplace transforms, respectively.

Denote the eigenvalues and the eigenfunctions of Δ_D by $\{\lambda_n, \phi_n\}_{n=1}^\infty$, where $\phi_n \in C^\infty(D)$. The corresponding heat kernel is given by

$$p_D(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$\begin{aligned} T_D(t)f(x) &= E_x[f(X_t)I(t < \tau_D(X))] = \int_D p_D(t, x, y)f(y)dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t} \phi_n(x) \bar{f}(n) \end{aligned}$$

solves the Heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \frac{\partial u(t, x)}{\partial t} &= \Delta u(t, x), \quad x \in D, \quad t > 0, \\ u(t, x) &= 0, \quad x \in \partial D, \\ u(0, x) &= f(x), \quad x \in D. \end{aligned}$$

Fractional diffusion in bounded domains

- ▶ Let $\beta \in (0, 1)$, $D_\infty = (0, \infty) \times D$ and define

$$\mathcal{H}_\Delta(D_\infty) \equiv \left\{ u : D_\infty \rightarrow \mathbb{R} : \frac{\partial}{\partial t} u, \frac{\partial^\beta}{\partial t^\beta} u, \Delta u \in C(D_\infty), \right. \\ \left. \left| \frac{\partial}{\partial t} u(t, x) \right| \leq g(x) t^{\beta-1}, g \in L^\infty(D), t > 0 \right\}.$$

- ▶ Let $0 < \gamma < 1$. Let D be a bounded domain with $\partial D \in C^{1,\gamma}$, and
- ▶ $T_D(t)$ be the killed semigroup of Brownian motion $\{X_t\}$ in D .
- ▶ Let E_t be the process inverse to a stable subordinator of index $\beta \in (0, 1)$ independent of $\{X_t\}$.
- ▶ Let $f \in D(\Delta_D) \cap C^1(\bar{D}) \cap C^2(D)$ for which the eigenfunction expansion (of Δf) with respect to the C.O.N. basis $\{\phi_n : n \in \mathbb{N}\}$ converges uniformly and absolutely.

Then the unique (classical) solution of

$$\begin{aligned}
 u &\in \mathcal{H}_\Delta(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D}) \\
 \frac{\partial^\beta}{\partial t^\beta} u(t, x) &= \Delta u(t, x); \quad x \in D, \quad t > 0 \\
 u(t, x) &= 0, \quad x \in \partial D, \quad t > 0, \\
 u(0, x) &= f(x), \quad x \in D.
 \end{aligned} \tag{8}$$

is given by

$$\begin{aligned}
 u(t, x) &= \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) E_{\beta}(-\lambda_n t^{\beta}) \\
 &= E_x[f(X(E_t)) I(\tau_D(X) > E_t)] \\
 &= E_x[f(X(E_t)) I(\tau_D(X(E)) > t)] \\
 &= \frac{t}{\beta} \int_0^{\infty} T_D(l) f(x) g_{\beta}(tl^{-1/\beta}) l^{-1/\beta-1} dl \\
 &= \int_0^{\infty} T_D((t/l)^{\beta}) f(x) g_{\beta}(l) dl.
 \end{aligned}$$

Joint work with Meerschaert and Vellaisamy (2008).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

Proof

Assume that $u(t, x)$ solves (8). Taking ϕ_n - transforms in (8) we obtain (using Green's second identity)

$$\frac{\partial^\beta}{\partial t^\beta} \bar{u}(t, n) = -\lambda_n \bar{u}(t, n). \quad (9)$$

taking Laplace transforms on both sides of (9), we get

$$s^\beta \hat{u}(s, n) - s^{\beta-1} \bar{u}(0, n) = -\lambda_n \hat{u}(s, n) \quad (10)$$

which leads to

$$\hat{u}(s, n) = \frac{\bar{f}(n) s^{\beta-1}}{s^\beta + \lambda_n}. \quad (11)$$

By inverting the above Laplace transform, we obtain

$$\bar{u}(t, n) = \bar{f}(n)E_{\beta}(-\lambda_n t^{\beta})$$

in terms of the Mittag-Leffler function defined by

$$E_{\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1 + \beta k)}.$$

Inverting now the ϕ_n -transform, we get an L^2 -convergent solution of Equation (8) as (for each $t \geq 0$)

$$u(t, x) = \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_{\beta}(-\lambda_n t^{\beta}) \quad (12)$$

To get the probabilistic form of the solution we proceed as before

$$\bar{u}(s, n) = \frac{\bar{f}(n)s^{\beta-1}}{s^{\beta} + \lambda_n} = \bar{f}(n)s^{\beta-1} \int_0^{\infty} \exp(-[s^{\beta} + \lambda_n]u) du$$

and observe that

$$e^{-u\lambda_n}\bar{f}(n) = \phi_n - \text{transform of } T_D f$$

Corollary

$$\|u(t, x)\|_{L^2} \sim C(x)E_\beta(-\lambda_1 t^\beta) \sim \frac{C(x)}{\lambda_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have
 $E_\beta(-\lambda_1 t^\beta) = e^{-\lambda_1 t}$ so

$$\|u(t, x)\|_{L^2} \sim C(x)E_1(-\lambda_1 t) = C(x)e^{-\lambda_1 t}$$

Extensions

Uniformly elliptic operator of divergence form is defined on C^2 functions by

$$Lu = \sum_{i,j=1}^d \frac{\partial (a_{ij}(x)(\partial u / \partial x_i))}{\partial x_j} \quad (13)$$

with $a_{ij}(x) = a_{ji}(x)$ and, for some $\lambda > 0$,

$$\lambda \sum_{i=1}^n y_i^2 \leq \sum_{i,j=1}^n a_{ij}(x) y_i y_j \leq \lambda^{-1} \sum_{i=1}^n y_i^2, \quad \forall y \in \mathbb{R}^d. \quad (14)$$

Work in progress for the Subordinated Brownian motions, e.g. symmetric stable process as the outer process....

IBM in bounded domains

Let $Z_t = X(|Y_t|)$ be the iterated Brownian motion,
 $D_\infty = (0, \infty) \times D$ and define

$$\mathcal{H}_{\Delta^2}(D_\infty) \equiv \left\{ u : D_\infty \rightarrow \mathbb{R} : \frac{\partial}{\partial t} u, \Delta^2 u \in C(D_\infty), \Delta u \in C^1(\bar{D}), \right. \\ \left. \left| \frac{\partial}{\partial t} u(t, x) \right| \leq g(x)t^{-1/2}, g \in L^\infty(D), t > 0 \right\}.$$

Let D be a domain with $\partial D \in C^{1,\gamma}$, $0 < \gamma < 1$. Let $\{X_t\}$ be Brownian motion in \mathbb{R}^d , and $\{Y_t\}$ be an independent Brownian motion in \mathbb{R} . Let $\{E_t\}$ be the process inverse to a stable subordinator of index $\beta = 1/2$ independent of $\{X_t\}$.

Let $f \in D(\Delta_D) \cap C^1(\bar{D}) \cap C^2(D) (\subset L^2(D))$ be such that the eigenfunction expansion of Δf with respect to $\{\phi_n : n \geq 1\}$ converges absolutely and uniformly. Then the (classical) solution of

$$\begin{aligned}
 u &\in \mathcal{H}_{\Delta^2}(D_\infty) \cap C_b(\bar{D}_\infty) \cap C^1(\bar{D}); \\
 \frac{\partial}{\partial t} u(t, x) &= \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t, x), \quad x \in D, \quad t > 0; \quad (15) \\
 u(t, x) &= \Delta u(t, x) = 0, \quad t \geq 0, \quad x \in \partial D; \\
 u(0, x) &= f(x), \quad x \in D
 \end{aligned}$$

is given by

$$\begin{aligned}
 u(t, x) &= E_x[f(Z_t)I(\tau_D(X) > |Y_t|)] \\
 &= E_x[f(X(E_t))I(\tau_D(X) > E_t)] \\
 &= E_x[f(X(E_t))I(\tau_D(X(E)) > t)] \\
 &= 2 \int_0^\infty T_D(l)f(x)p(t, l)dl, \tag{16}
 \end{aligned}$$

where $T_D(l)$ is the heat semigroup in D , and $p(t, l)$ is the transition density of one-dimensional Brownian motion $\{Y_t\}$. Proof uses again equivalence with fractional Cauchy problem for $\beta = 1/2$.