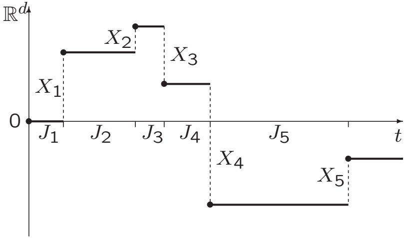
Cauchy Problems Solved by Running Subordinate Processes

Informs 2010

Erkan Nane
DEPARTMENT OF MATHEMATICS AND STATISTICS
AUBURN UNIVERSITY

August 18, 2010

Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

J_n's are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the nth jump.
$N(t) = \max\{n \geq 0 : \ T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a β-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \ \ D(u) > t\}.$$

The self-similar limit $E(ct) \stackrel{d}{=} c^{\beta} E(t)$ is non-Markovian.

Continuous time random walks (CTRW)

$S(n) = X_1 + \cdots + X_n$ particle location after n jumps
has scaling limit $c^{-1/2}S([ct]) \implies B(t)$ a Brownian motion.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies E(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

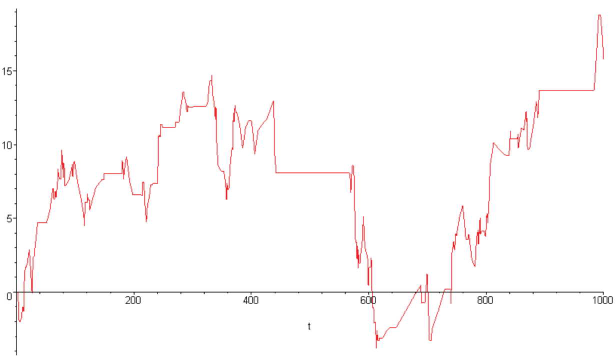
$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}.$$

$S(N(t))$ is particle location at time $t > 0$.
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^\beta)^{-1/2}S(c^\beta \cdot c^{-\beta}N(ct)) \\ &\approx (c^\beta)^{-1/2}S(c^\beta E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \stackrel{d}{=} c^{\beta/2}B(E(t))$ is non-Markovian.

Scaling limit: Subordinated motion



Limit retains long waiting times.

Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

CTRW with serial dependence

Particle jumps $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with Z_n IID.

Short range dependence: $\sum_{n=1}^{\infty} |\mathbb{E}(X_n X_0)| < \infty \implies$ the usual limit and PDE.

Long range dependence: If Z_n has light tails, subordinated fractional Brownian motion limit $B_H(E(t))$. Hahn, Kobayashi and Umarov (2010) established a governing equation.

For heavy tails, subordinated linear fractional stable motion $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Due to Meerschaert, Nane and Xiao (2009).

Fractional time derivative: Two approaches

- Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^{\beta}} \right]$$

with Laplace transform $s^{\beta}\tilde{g}(s)$, $\tilde{g}(s) = \int_0^{\infty} e^{-st} g(t) dt$ denotes the usual Laplace transform of g.

- Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^{\beta}} \qquad (1)$$

was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^{\beta} g(t)$ is $s^{\beta}\tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^{\beta}(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^{\beta}(e^{\lambda t}) = \lambda^{\beta} e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^{\beta}(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Diffusion

Let L_x be the generator of some continuous Markov process $X(t)$. Then $p(t,x) = \mathbb{E}_x[f(X(t))]$ is the unique solution of the heat-type Cauchy problem

$$\partial_t p(t,x) = L_x p(t,x),\ \ t > 0, x \in \mathbb{R}^d; \quad p(0,x) = f(x),\ \ x \in \mathbb{R}^d$$

Examples:
X: Brownian motion then $L_x = \Delta_x$, BM is a stochastic solution of the heat equation
X: Symmetric α-stable process then $L_x = -(-\Delta)^{\alpha/2}$

Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{2}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(2) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(X(E(t)))] = \int_0^\infty p(s,x) g_{E(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(X(t))]$ and $E(t) = \inf\{\tau > 0 : \ D_\tau > t\}$, $D(t)$ is a stable subordinator with index β and $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx xt^{\beta/2}$.

Equivalence to higher order PDE's

- For any $m = 2, 3, 4, \dots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t,x); \quad u(0,x) = f(x) \tag{3}$$

  and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \tag{4}$$

  have the same unique solution given by

$$u(t,x) = \mathbb{E}_x[f(X(E^{1/m}(t)))] = \int_0^\infty p(s,x) g_{E^{1/m}(t)}(s)\, ds$$

- Due to Allouba and Zheng (2001), Baeumer, Meerschaert, and Nane (2007), Keyantuo and Lizama (2009), Li et al. (2010).

Connections to iterated Brownian motions

- Orsingher and Beghin (2004, 2008) show that the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{5}$$

is given by running

$$I_{n+1}(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where B_j's are independent Brownian motions, i.e., $u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (5), and solves (3) for $m = 2^n$.

Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of Δ on a bounded domain D with Dirichlet boundary conditions by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta\phi_n(x) = -\mu_n\phi_n(x), \quad x \in D; \phi_n(x) = 0, \quad x \in \partial D.$$

$\tau_D(X) = \inf\{t \geq 0 : \ X(t) \notin D\}$ is the first exit time of a process X, and let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \ = \ \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

Fractional diffusion in bounded domains

$$\begin{aligned}
\partial_t^\beta u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \qquad (6)\\
u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.
\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)M_\beta(-\mu_n t^\beta) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl\\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))]\\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)]
\end{aligned}$$

$\boxed{\text{Joint work with Meerschaert and Vellaisamy (2009).}}$

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

Sketch of Proof

- Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta_x u(t,x)dx = \int_D u(t,x)\Delta\phi_n(x)$$
$$= -\mu_n \int_D u(t,x)\phi_n(x)dx = -\mu_n \bar{u}(t,n)$$

Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\mu_n \bar{u}(t,n). \tag{7}$$

- taking Laplace transforms on both sides of (7), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\mu_n \hat{u}(s,n) \tag{8}$$

- Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\mu_n}$.

Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n) M_\beta(-\mu_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\mu E(t)}) = \int_0^\infty e^{-\mu l} g_{E(t)}(l)\,dl = M_\beta(-\mu t^\beta).$$

Inverting now the ϕ_n-transform, we get an L^2-convergent solution of Equation (6) as (for each $t \geq 0$)

$$u(t,x) = \sum^{\infty} \bar{f}(n) \phi_n(x) M_\beta(-\mu_n t^\beta) \tag{9}$$

Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)M_\beta(-\mu_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)\int_0^\infty e^{-\mu_n l} g_{E(t)}(l)dl \\
&= \int_0^\infty \left(\sum_{n=1}^{\infty} \bar{f}(n)e^{-\mu_n l}\phi_n(x)\right) g_{E(t)}(l)dl \qquad (10) \\
&= \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\
&= \int_0^\infty \mathbb{E}_x[f(B(l))I(\tau_D > l)]g_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))]
\end{aligned}$$

IBM in bounded domains

The (classical) solution of

$$\begin{aligned}
\partial_t u(t,x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^{2^n} u(t,x), \quad x \in D, \ t > 0; \\
u(t,x) &= \Delta_x^l u(t,x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1; \\
u(0,x) &= f(x), \ x \in D
\end{aligned}$$

is given by (running
$I_{n+1}(t) = B_1(|B_2(\cdots|B_{n+1}(t)|)|) = B_1(|I_n(t)|))$

$$\begin{aligned}
u(t,x) &= E_x[f(I_{n+1}(t))I(\tau_D(B_1) > |I_n(t)|)] \\
&= 2\int_0^\infty T_D(l)f(x)h(t,l)dl, \qquad (11)
\end{aligned}$$

where $h(t,l)$ is the transition density of $\{I_n(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$.

Corollary

$$||u(t,\cdot)||_{L^2} \sim CE_\beta(-\mu_1 t^\beta) \sim \frac{C}{\mu_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have $E_\beta(-\mu_1 t^\beta) = e^{-\mu_1 t}$ so

$$||u(t,\cdot)||_{L^2} \sim CE_1(-\mu_1 t) = Ce^{-\mu_1 t}$$

Stochastic model for ultraslow diffusion

Let $supp\nu \subset (0,1)$ be a finite measure.
$W(t)$, increasing Lévy process with laplace transform

$$\mathbb{E}[e^{-sW(t)}] = e^{-t\left[\int_0^\infty (e^{-sx}-1)\phi_W(dx)\right]} = e^{-t\left[\int_0^1 s^\beta \Gamma(1-\beta)\nu(d\beta)\right]}.$$

The Lévy measure is

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\nu(d\beta), \tag{12}$$

Let $E(t) = \inf\{\tau \geq 0 : \ W(\tau) \geq t\}$ be the inverse process.
Then stochastic model for **ultraslow** diffusion $B(E(t))$ is a stochastic solution of

$$\mathbb{D}^{(\nu)}u(t,x) := \int_0^1 \partial_t^\beta u(t,x)\Gamma(1-\beta)\nu(d\beta) = \Delta_x u(t,x)$$

For special ν: $\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx x(\log t)^{\beta/2}$.

Ultraslow diffusion in bounded domains

$$\begin{aligned} \mathbb{D}^{(\nu)}u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h(t,\mu_n) = \int_0^{\infty} T_D(l)f(x)g_{E(t)}(l)dl \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \end{aligned}$$

where $h(t,\mu_n) = \mathbb{E}(e^{-\mu_n E(t)})$.

Joint work with Meerschaert and Vellaisamy (2009).

Eigenvalue problem for distributed order time derivative

$h(t,\mu) = \mathbb{E}(e^{-\mu E(t)})$ is the solution of

$$\mathbb{D}^{(\nu)} h(t,\mu) = -\mu h(t,\mu); \quad h(0,\mu) = 1. \tag{13}$$

In the case $\nu(d\beta) = p(\beta)d\beta$; by inverse Laplace transform it has the representation

$$h(t,\mu) = \frac{\mu}{\pi} \int_0^\infty r^{-1} e^{-tr} \Phi(r,1) dr \tag{14}$$

where for $U(r) = \int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta$

$$\Phi(r,1) = \frac{U(r)}{[\int_0^1 r^\beta \cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \mu]^2 + [U(r)]^2}.$$

Due to Kochubei (2008).

Tempered fractional model

$D_\lambda(t)$, increasing Lévy process with laplace transform

$$\mathbb{E}[e^{-sD_\lambda(t)}] = e^{-t\psi_\lambda(s)} = e^{-t\left[\int_0^\infty (e^{-sx}-1)\phi_{D_\lambda}(dx)\right]} = e^{-t\left[(s+\lambda)^\beta-\lambda^\beta\right]}.$$

The Lévy measure is

$$\phi_{D_\lambda}(t,\infty) = \frac{1}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr, \tag{15}$$

Let $E_\lambda(t) = \inf\{\tau \geq 0 : \ D_\lambda(\tau) \geq t\}$ be the inverse process. Define the caputo tempered fractional time derivative by

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t)g(t) - g(0)\phi_{D_\lambda}(t,\infty)\\ &= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t \frac{e^{\lambda s}g(s)\,ds}{(t-s)^\beta} - \lambda^\beta g(t) \\ &\quad - g(0)\phi_{D_\lambda}(t,\infty)\end{aligned} \tag{16}$$

Tempered fractional cauchy problem

Then $\mathbb{E}[f(B(E_\lambda(t)))]$ is a stochastic solution of

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} u(t,x) = \Delta_x u(t,x);\ \ u(t,0) = f(x).$$

Due to Meerschaert and Sheffler (2008).

$$\mathbb{E}_x(B(E_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$B(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

Tempered fractional cauchy problem in bounded domains

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} u(t,x) &= \Delta_x u(t,x), \quad x\in D,\ t>0;\\ u(t,x) &= 0,\ x\in\partial D,\ t>0;\quad u(0,x)=f(x),\ x\in D.\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned}u(t,x) &= \mathbb{E}_x(f(B(E_\lambda(t)))I(\tau_{D_\lambda}(B)>E_\lambda(t))\\ &= \int_0^\infty T_D(l)f(x)g_\lambda(t,l)dl\\ &= \sum_{n=1}^\infty \bar{f}(n)\phi_n(x)h_\lambda(t,\mu_n),\end{aligned}$$

where $h_\lambda(t,\mu)=\mathbb{E}(e^{-\mu E_\lambda(t)})=\int_0^\infty e^{-\mu x}g_\lambda(t,x)\,dx$ is the Laplace transform of $E_\lambda(t)$ (Meerschaert, Nane and Vellaisamy 2010).

Eigenvalue problem for tempered fractional time derivative

$h_\lambda(t,\mu) = \mathbb{E}(e^{-\mu E_\lambda(t)})$ is the unique solution of

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} h_\lambda(t,\mu) = -\mu h_\lambda(t,\mu); \quad h_\lambda(0,\mu) = 1. \tag{17}$$

For any $\mu, \lambda > 0$, $\mu \neq \lambda^\beta$, the function $h_\lambda(t,\mu)$ has the representation

$$h_\lambda(t,\mu) = \frac{\mu}{\pi} \int_0^\infty (r+\lambda)^{-1} e^{-t(r+\lambda)} \Phi(r,1)\,dr, \tag{18}$$

where

$$\Phi(r,1) = \frac{r^\beta \sin(\beta\pi)}{r^{2\beta}\sin^2(\beta\pi) + (\mu - \lambda^\beta + r^\beta \cos(\beta\pi))^2}$$

Due to Meerschaert, Nane and Vellaisamy (2010).

Other Subordinators; Cauchy process

$X(t)$ a continuous Markov process with generator $\mathcal{A}$, $Y(t)$ be a Cauchy process independent of $X(t)$. Then $u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= -\frac{2\mathcal{A}f(x)}{\pi t} - \mathcal{A}^2 u(t,x), \ t>0, \quad x \in \mathbb{R}^d; \\ u(0,x) &= f(x) \ x \in \mathbb{R}^d. \end{aligned}$$

Due to Nane (2008).
Proof uses the fact that the density $p(t,s)$ of $Y(t)$ solves

$$(\partial_s^2 + \partial_t^2)p(t,s) = 0.$$

Nonhomogeneous wave equation

This reduces to **nonhomogeneous wave equation** in the case X is another Cauchy process independent of Y,
The generator of X is $\mathcal{A} = -(-\Delta)^{1/2}$, fractional Laplacian.
$u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$
\begin{aligned}
\partial_t^2 u(t,x) &= \frac{2(-\Delta)^{1/2} f(x)}{\pi t} + \Delta u(t,x), \; t>0, \quad x \in \mathbb{R}^d; \\
u(0,x) &= f(x), \; x \in \mathbb{R}^d
\end{aligned}
$$

This is one of the most interesting PDE connections of these iterated processes.

Bounded domains

Let D be a bounded domain. Then

$$\begin{aligned} u(t,x) &= \mathbb{E}_x[f(B(|Y(t)|)I(\tau_D(B) > |Y(t)|)] \\ &= 2\int_0^\infty \left(\sum_{n=1}^\infty e^{-\mu_n s}\bar{f}(n)\phi_n(x)\right)\frac{t}{\pi(t^2+s^2)}ds \end{aligned}$$

is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= -\frac{2\Delta f(x)}{\pi t} - \Delta^2 u(t,x), \quad t>0, \quad x \in D \quad (19) \\ u(0,x) &= f(x), \quad x \in D, \\ u(t,x) &= \Delta u(t,x) = 0, \quad x \in \partial D, \ t>0. \end{aligned}$$

Due to Nane (2010).

Stable subordinators: $\alpha \neq 1$

$\alpha \in (0,2)$ be rational $\alpha = l/m$, where l and m are relatively prime.
$Y(t)$; a symmetric α-stable process
Then $u(t,x) = \mathbb{E}_x[f(B(|Y(t)|)]$ is a solution of

$$\begin{aligned}
(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) &= -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p^{\alpha}(t,s)\Big|_{s=0}\right)\Delta^{2i-1}f(x) \\
&\quad - \Delta^{2l}u(t,x), \qquad t>0, \quad x\in\mathbb{R}^d \\
u(0,x) &= f(x), \qquad x\in\mathbb{R}^d.
\end{aligned}$$

Where $p^{\alpha}(t,s)$ is the transition density of symmetric α-stable process $Y(t)$. The proof uses the fact that

$$\partial_t^{2m}p^{\alpha}(t,x) = (-\partial_x^2)^l p^{\alpha}(t,x)$$

Bounded domains

$\alpha \in (0,2)$ a rational $\alpha = l/m$, where l and m are relatively prime.
D a bounded domain.
Then $u(t,x) = \mathbb{E}_x[f(B(|Y(t)|)I(\tau_D(B) > |Y(t)|)]$ is a classical solution of

$$
\begin{aligned}
(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) &= -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p^{\alpha}(t,s)\Big|_{s=0}\right)\Delta^{2i-1}f(x) \\
&\quad - \Delta^{2l}u(t,x), \qquad t>0, \quad x \in D \\
u(0,x) &= f(x), \qquad x \in D. \\
u(t,x) &= \Delta^{j}u(t,x) = 0, \quad x \in \partial D, \ t>0, \ j = 1, \cdots, 2l
\end{aligned}
$$

Further research

- Work in progress for the **subordinated Brownian motions**, e.g. symmetric stable process as the outer process. The corresponding space operators are $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$
- Extension to Neumann boundary conditions...
- Fractal properties of $B(E(t))$ and other subordinate processes
- Applications-interdisciplinary research

Thank You!