Cauchy Problems in Bounded Domains and Iterated Processes

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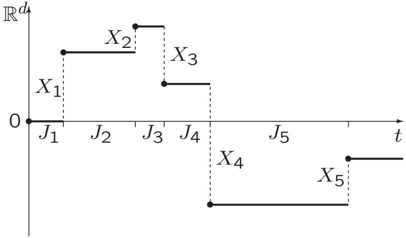
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

J_n's are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the nth jump.
$N(t) = \max\{n \geq 0 : \; T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a β-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \quad D(u) > t\}.$$

The self-similar limit $E(ct) \overset{d}{=} c^{\beta} E(t)$ is non-Markovian.

Continuous time random walks (CTRW)

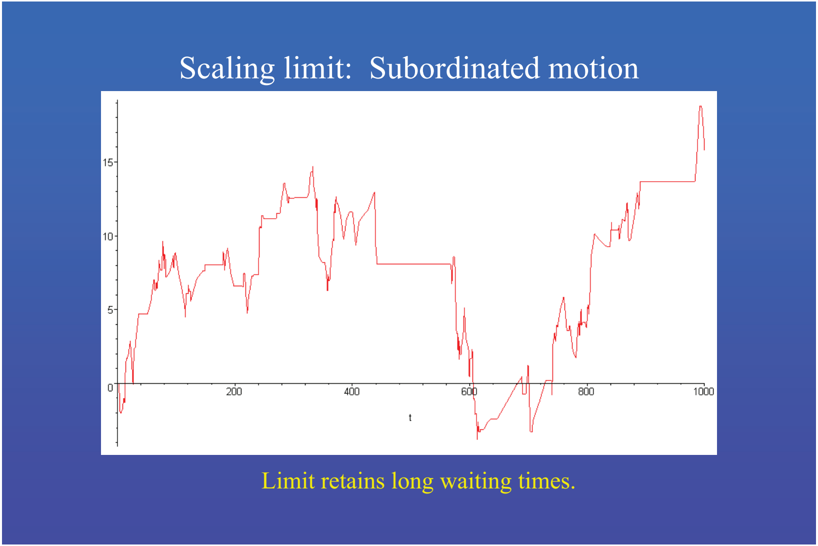
$S(n) = X_1 + \cdots + X_n$ particle location after n jumps
has scaling limit $c^{-1/2} S([ct]) \implies B(t)$ a Brownian motion.
Number of jumps has scaling limit $c^{-\beta} N(ct) \implies E(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}.$$

$S(N(t))$ is particle location at time $t > 0$.
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2} S(N(ct)) &= (c^{\beta})^{-1/2} S(c^{\beta} \cdot c^{-\beta} N(ct)) \\ &\approx (c^{\beta})^{-1/2} S(c^{\beta} E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \overset{d}{=} c^{\beta/2} B(E(t))$ is non-Markovian.



Scaling limit: Subordinated motion

Limit retains long waiting times.

Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

CTRW with serial dependence

Particle jumps $X_n = \sum_{j=0}^{\infty} Z_{n-j}$ with Z_n IID.

Short range dependence $\Longrightarrow$ the usual limit and PDE.

Long range dependence: If Z_n has light tails, subordinated fractional Brownian motion limit $B_H(E(t))$.

For heavy tails, subordinated linear fractional stable motion $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Due to Meerschaert, Nane and Xiao (2009).

Fractional time derivative: Two approaches

- Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^{\beta}} \right]$$

  with laplace transform $s^{\beta}\tilde{g}(s)$, $\tilde{g}(s) = \int_0^{\infty} e^{-st} g(t) dt$ denotes the usual Laplace transform of g.

- Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^{\beta} g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^{\beta}} \tag{1}$$

  was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^{\beta} g(t)$ is $s^{\beta}\tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

- $$D_t^{\beta}(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^{\beta}(e^{\lambda t}) = \lambda^{\beta} e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)} ?$$
- $$D_t^{\beta}(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Time fractional model for Anamolous sub-diffusion

Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = Lu(t,x); \quad u(0,x) = f(x) \tag{2}$$

where $0 < \beta < 1$ and L is the generator of some continuous Markov process $X(t)$.
Zaslavsky (1994) used this to model Hamiltonian chaos.
When $p(t,x) = T(t)f(x)$ is a solution of $\partial_t p(t,x) = Lp(t,x)$(with corresponding process X_t, $E(t) = \inf\{u : \ D_u > t\}$, $D(t)$ is a stable subordinator with index β) , the formula

$$u(t,x) \;=\; \mathbb{E}_x\Big[f(X(E(t)))\Big] = \int_0^\infty p(s,x) g_{E(t)}(s)ds$$

yields a solution of (2). Due to Baeumer and Meerschaert (2002).
$\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx xt^{\beta/2}$.

Equivalence to higher order PDE's

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t,x); \quad u(0,x) = f(x) \tag{3}$$

  and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \tag{4}$$

  have the same unique solution given by

$$u(t,x) = \int_0^\infty p(s,x) g_{E^{1/m}(t)}(s)\, ds$$

- Due to Baeumer, Meerschaert, and Nane (2007).

Connections to iterated Brownian motions

- Orsingher and Benghin (2004, 2008) show that the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{5}$$

is given by running

$$I_{n+1}(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where B_j's are independent Brownian motions, i.e., $u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (5), and solves (3) for $m = 2^n$.

Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of Δ on a bounded domain D with Diriclet boundary conditions by $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$; $\Delta \phi_n = -\lambda_n \phi_n$. Define the first exit time of a process X to be $\tau_D(X) = \inf\{t \geq 0 : \ X(t) \notin D\}$. Let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \;=\; \mathbb{E}_x[f(B(t))I(\tau_D > t)] = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned}
\partial_t u(t,x) &= \Delta u(t,x), \quad x \in D,\ t > 0,\\
u(t,x) &= 0, \quad x \in \partial D,\\
u(0,x) &= f(x), \quad x \in D.
\end{aligned}$$

Fractional diffusion in bounded domains

$$\begin{aligned} \partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D. \end{aligned} \tag{6}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)] \end{aligned}$$

Joint work with Meerschaert and Vellaisamy (2009).

Analytic solution in intervals $(0,M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

Sketch of Proof

- Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta u(t,x)dx = \int_D u(t,x)\Delta\phi_n(x)$$
$$= -\lambda_n \int_D u(t,x)\phi_n(x)dx = -\lambda_n \bar{u}(t,n)$$

  Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\lambda_n \bar{u}(t,n). \tag{7}$$

- taking Laplace transforms on both sides of (7), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\lambda_n \hat{u}(s,n) \tag{8}$$

- Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\lambda_n}$.

Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n) E_\beta(-\lambda_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\lambda E(t)}) = \int_0^\infty e^{-\lambda u} g_{E(t)}(u) du = E_\beta(-\lambda t^\beta).$$

Inverting now the ϕ_n-transform, we get an L^2-convergent solution of Equation (6) as (for each $t \geq 0$)

$$u(t,x) = \sum^{\infty} \bar{f}(n) \phi_n(x) E_\beta(-\lambda_n t^\beta) \qquad (9)$$

Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)E_\beta(-\lambda_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)\int_0^\infty e^{-\lambda_n u} g_{E(t)}(u)du \\
&= \int_0^\infty \left(\sum_{n=1}^{\infty} \bar{f}(n)e^{-\lambda_n u}\phi_n(x)\right) g_{E(t)}(u)du \qquad (10) \\
&= \int_0^\infty T_D(u)f(x)g_{E(t)}(u)du \\
&= \int_0^\infty \mathbb{E}_x[f(B(u))I(\tau_D > u)]g_{E(t)}(u)du \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))]
\end{aligned}
$$

IBM in bounded domains

The (classical) solution of

$$\begin{aligned} \partial_t u(t,x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^{2^n} u(t,x), \quad x \in D, \ t > 0; \qquad (11) \\ u(t,x) &= \Delta^l u(t,x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1; \\ u(0,x) &= f(x), \ x \in D \end{aligned}$$

is given by (running

$I_{n+1}(t) = B_1(|B_2(\cdots|B_{n+1}(t)|)|) = B_1(|I_n(t)|))$

$$\begin{aligned} u(t,x) &= E_x[f(I_{n+1}(t)) I(\tau_D(B_1) > |I_n(t)|)] \\ &= 2\int_0^\infty T_D(l) f(x) h(t,l) dl, \qquad (12) \end{aligned}$$

where $h(t,l)$ is the transition density of $\{I_n(t)\}$.

Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$.

Corollary

$$||u(t,x)||_{L^2} \sim C(x)E_\beta(-\lambda_1 t^\beta) \sim \frac{C(x)}{\lambda_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have $E_\beta(-\lambda_1 t^\beta) = e^{-\lambda_1 t}$ so

$$||u(t,x)||_{L^2} \sim C(x)E_1(-\lambda_1 t) = C(x)e^{-\lambda_1 t}$$

Stochastic model for ultraslow diffusion

Let $supp\mu \subset (0,1)$ be a finite measure. B_i iid with dist. μ.
$J_i^c \overset{d}{=} c^{-1/\beta} J_1$ nonnegative iid with
$P(J_1 > t) = \int_0^1 t^{-\beta}\mu(d\beta), \quad t \geq 1$ and

$$P(c^{-1/\beta}J_1 > u | B_1 = \beta) = \begin{cases} 1, & 0 \leq u < c^{-1/\beta} \\ c^{-1}u^{-\beta}, & u \geq c^{-1/\beta} \end{cases}$$

Time of the nth jump at scale c has
$T^{(ct)}(t) = \sum_{i=1}^{[ct]} J_i^c \implies W(t)$, increasing Lévy process.
The number of jumps by time $t \geq 0$ at scale c has scaling limit;
$N_t^c = \max\{n \geq 0 : \ T^c(n) \leq t\} \implies E(t) = \inf\{\tau \geq 0 : \ W(\tau) \geq t\}$.
X_i^c iid jumps has scaling limit
$S^c(ct) = X_1^c + X_2^c + \cdots + X_{[ct]}^c \implies B(t)$ then

$$S^c(N_t^c) \implies B(E(t))$$

Ultraslow diffusion

$\mathbb{E}[e^{-sW(t)}] = e^{-t\psi_W(s)}$ and Laplace exponent

$$\psi_W(s) = \int_0^\infty (e^{-sx} - 1)\phi_W(dx) = \int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta). \quad (13)$$

The Lévy measure is

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\mu(d\beta), \quad (14)$$

Then stochastic model for **ultraslow** diffusion $B(E(t))$ is a stochastic solution of

$$\mathbb{D}^{(\nu)}u(t,x) := \int_0^1 \partial_t^\beta u(t,x)\nu(d\beta) = \Delta u(t,x); \quad \nu(d\beta) = \Gamma(1-\beta)\mu(d\beta).$$

For special ν: $\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx x(\log t)^{\beta/2}$.

Ultraslow diffusion in bounded domains

$$\begin{aligned}
\mathbb{D}^{(\nu)}u(t,x) &= \Delta u(t,x); \quad x \in D, \ t > 0 \\
u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.
\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h(t,\lambda_n) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(X(E(t)))I(\tau_D(X) > E(t))] \\
&= \mathbb{E}_x[f(X(E(t)))I(\tau_D(X(E)) > t)]
\end{aligned}$$

where $h(t,\lambda_n) = \mathbb{E}(e^{-\lambda_n E(t)})$.

Joint work with Meerschaert and Vellaisamy (2009).

Eigenvalue problem for distributed order time derivative

$h(t,\lambda)=\mathbb{E}(e^{-\lambda E(t)})$ is the solution of

$$\mathbb{D}^{(\nu)}h(t,\lambda)=-\lambda h(t,\lambda);\quad h(0,\lambda)=1. \tag{15}$$

In the case $\mu(d\beta)=p(\beta)d\beta$; by inverse laplace transform it has the representation

$$h(t,\lambda)=\frac{\lambda}{\pi}\int_0^\infty r^{-1}e^{-tr}\Phi(r,1)dr \tag{16}$$

where for $U(r)=\int_0^1 r^\beta\sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta$

$$\Phi(r,1)=\frac{U(r)}{[\int_0^1 r^\beta\cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta+\lambda]^2+[U(r)]^2}.$$

Due to Kochubei (2008).

Extensions: Markov generators

The Markov process $X(t)$ with generator

$$Lu = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{x_i}\partial_{x_j}u + \sum_{i=1}^{d} b_i(x)\partial_{x_i}u \tag{17}$$

solves $dX(t) = \sigma(X(t))dB(t) + b(X(t))dt$ with $a = \sigma\sigma^T$. Then

$$p(t,x) = T_D(t)f(x) = \mathbb{E}[f(X(t))I(\tau_D(X) > t)]$$

solves the Cauchy problem

$$\partial_t p(t,x) = Lp(t,x)$$

with Dirichlet boundary conditions.

Further research

- Work in progress for the **subordinated Brownian motions**, e.g. symmetric stable process as the outer process. The corresponding space operators are $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$
- Extension to Neumann boundary conditions...
- Extensions to other time operators; tempered fractional derivative...
- Fractal properties of $B(E(t))$ and other subordinate processes
- Applications-interdisciplinary research

Thank You!