# TIME-CHANGED PROCESSES AND CAUCHY PROBLEMS

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# Outline

# Probability and transforms

If the random variable $Y$ has density $f(x)$ so that

$$P(a \le Y \le b) = \int_a^b f(x)dx$$

then $f(x)$ has Fourier transform

$$\begin{aligned}\hat{f}(k) &= E(e^{-ikY}) = \int_{-\infty}^{\infty} e^{-ikx} f(x)dx \\ &= \int_{-\infty}^{\infty} (1 - ikx + \frac{1}{2!}(ikx)^2 + \cdots) f(x)dx \\ &= 1 - ik\mu_1 - \frac{1}{2!}k^2\mu_2 + \cdots\end{aligned}$$

where the $l$th moment is $\mu_l = \int_{-\infty}^{\infty} x^l f(x)dx$

# Central limit theorem

If $\mu_1 = 0$ and $\mu_2 = 2$ then $\hat{f}(k) = 1 - k^2 + \cdots$
The IID sum $S(n) = Y_1 + \cdots + Y_n$ has FT $\hat{f}(k)^n$ and the normalized sum $S(n)/\sqrt{n}$ has FT

$$\left(\hat{f}(k/\sqrt{n})\right)^n = (1 - (k/\sqrt{n})^2 + \cdots)^n$$
$$= \left(1 - \frac{k^2}{n} + \cdots\right)^n$$
$$\to e^{-k^2} \equiv \hat{g}(k) \quad \text{as} \quad n \to \infty.$$

Inverting the Fourier transform reveals a Gaussian(Normal) density

$$g(x) = \frac{1}{\sqrt{4\pi}} e^{-x^2/4}$$

# Brownian motion

If $Y_n$ represents a particle jump at time $n$ then $S(n) = Y_1 + \cdots + Y_n$ is the location of the particle at time $n$. Expanding the time scale by a factor of $c > 0$ and taking limits as $c \to \infty$ shows that $c^{-1/2} S([ct]) \implies W(t)$ since

$$\hat{f}(c^{-1/2}k)^{[ct]} = \left(1 - \frac{k^2}{c} + \cdots\right)^{[ct]}$$
$$\to e^{-k^2 t} \equiv \hat{p}(t,k) \quad \text{as} \quad c \to \infty$$

for all $t > 0$. Inverting the FT shows that the density of the limiting Brownian motion process $W(t)$ is Gaussian (Normal)

$$p(t,x) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/4t}.$$

# Brownian motion

## Classical random walk

$$S(t) = Y_1 + \cdots + Y_{[t]}$$

A particle takes a random jump $Y_n$ at time $t = n$. Particle location at time $t$ is a simple random walk $S(t)$ and scaling limit is a Brownian motion.

$$c^{-1/2} S(ct) \Rightarrow W(t) \approx \underbrace{N(0, \sigma^2 t)} \qquad (c \to \infty)$$

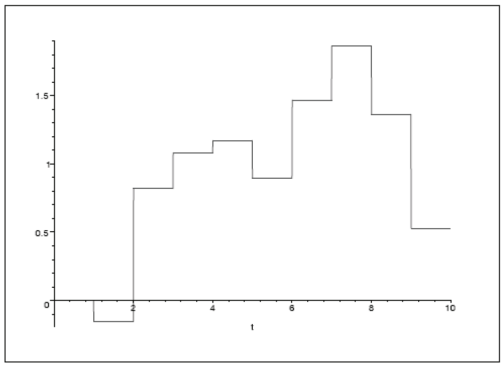
Contract spatial scale

Expand time scale

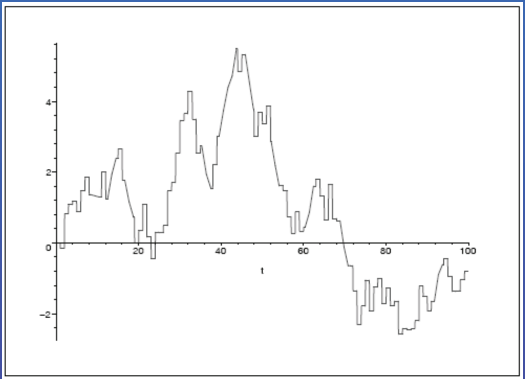
Normal limit density

Add an advective drift: $L(t) = vt + W(t) \approx N(vt, \sigma^2 t)$

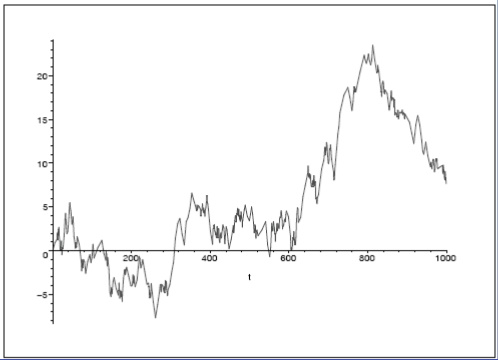
# Random walk simulation



# Longer time scale



# Scaling limit: Brownian motion




Random graph of fractal dimension 1.5 and no jumps.


Most likely shape of a Brownian path.

Microsoft stock-the last two years

# Some history of Brownian motion (BM)

- Robert Brown (1827), a Botanist: was first to observe that pollen grains in water move continuously and very erratically.
- Louis Bahcelier (1900): presented a stochastic analysis of the stock and option markets using BM
- Albert Einstein (1905): used BM to determine the law of the position of the particle...
- Norbert Wiener (1923): Mathematical foundations of BM
- Doob (1956): connections to analysis, heat equation
- Kolmogorov, Lévy, Khintchine, .....

# Derivatives and transforms

- If the Laplace transform of $f(t)$ is defined for $s > 0$ by

$$\tilde{f}(s) = \int_0^\infty e^{-st} f(t) dt$$

then $d_t f(t)$ has Laplace transform $s\tilde{f}(s) - f(0)$.

- If the Fourier transform of $f(x)$ is defined for $k \in \mathbb{R}$ by

$$\hat{f}(k) = \int_{\mathbb{R}} e^{-ikx} f(x) dx$$

then $d_x f(x)$ has Fourier transform $ik\hat{f}(k)$.

# The diffusion (heat) equation

Taking Fourier transforms in the classical diffusion equation

$$\partial_t p(t,x) = \partial_x^2 p(t,x)$$

yields

$$\partial_t \hat{p}(t,k) = (ik)^2 \hat{p}(t,k) = -k^2 \hat{p}(t,k)$$

whose solution

$$\hat{p}(t,k) = e^{-k^2 t}$$

inverts to the same limit density for the Brownian motion $W(t)$. For a cloud of diffusing particles $p(t,x)$ is the particle density.

Let $W_t \in \mathbb{R}$ be Brownian motion started at $x$. Then the function (convolution of f and $p(t,x)$)

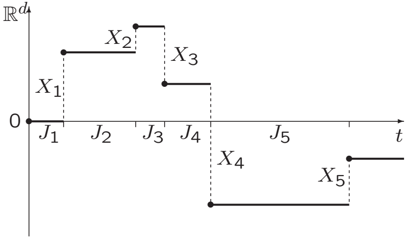
$$u(t,x) = E_x[f(W(t))] = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t} f(y)dy$$

solves the heat equation

$$\begin{aligned} \partial_t u(t,x) &= \partial_x^2 u(t,x), \qquad t>0, \quad x \in \mathbb{R} \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}. \end{aligned}$$

This is due to J.L. Doob (1956).
In this case we say, Brownian motion $W(t)$ is a stochastic solution of the heat equation.

## Continuous time random walks




The CTRW is a random walk with jumps $X_n$ separated by random waiting times $J_n$. The random vectors $(X_n, J_n)$ are i.i.d.

## Heavy tailed waiting times

Random wait $J_n$ between jumps, $n$th jump time given by a random walk

$$T(n) = J_1 + \cdots + J_n$$

Number of jumps by time $t$ is inverse $N(t) \geq n \longleftrightarrow T(n) \leq t$

For heavy tail waiting times $P(J_n > t) \approx Ct^{-\beta} \quad (0 < \beta < 1)$

$$c^{-1/\beta} T(ct) \Rightarrow P(t) \quad \Longleftrightarrow \quad c^{-\beta} N(ct) \Rightarrow Q(t)$$

Inverse processes have inverse scaling

$$P(ct) \approx c^{1/\beta} P(t) \quad \Longleftrightarrow \quad Q(ct) \approx c^{\beta} Q(t)$$

# Continuous time random walks (CTRW)

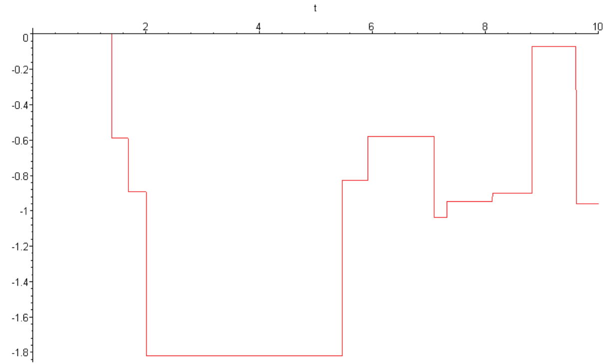
Particle jump random walk has scaling limit
$c^{-1/2}S([ct]) \implies W(t)$.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies Q(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}$$

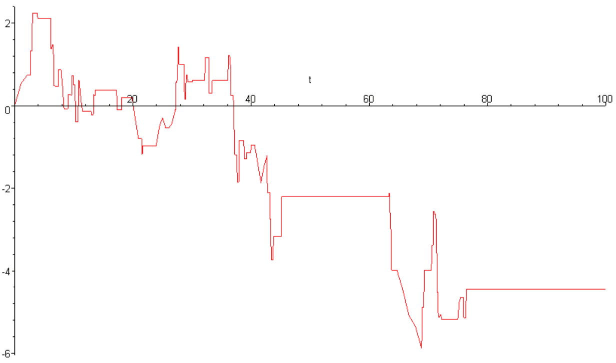
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta}\cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}Q(t)) \implies W(Q(t)). \end{aligned}$$

## CTRW simulation with heavy tail waiting times



# Longer time scale



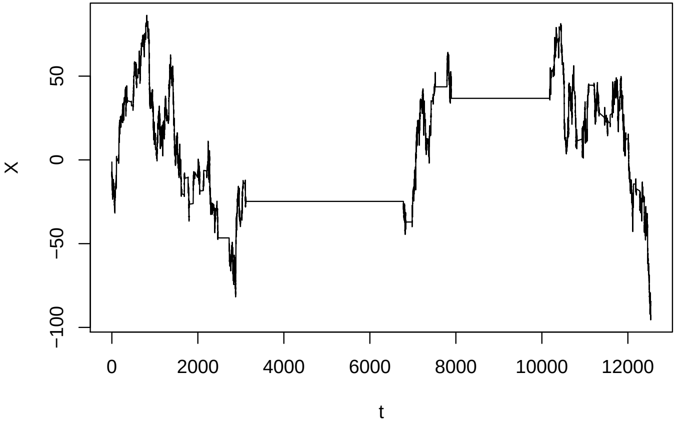



Figure: Typical sample path of the iterated process $W(Q(t))$. Here $W(t)$ is a Brownian motion and $Q(t)$ is the inverse of a $\beta = 0.8$-stable subordinator. Graph has dimension $1 + \beta/2 = 1 + 0.4$. The limit process retains long resting times


S&P 500 INDEX,RTH
15:58 01:10:2009
^GSPC
1070
1065
1060
1055
1050
1045
1040
1035
1030
1025
25.09
28.09
29.09
30.09
1.10
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# Power law waiting times

- Wait between solar flares $1 < \beta < 2$
- Wait between raindrops $\beta = 0.68$
- Wait between money transactions $\beta = 0.6$
- Wait between emails $\beta \approx 1.0$
- Wait between doctor visits $\beta \approx 1.4$
- Wait between earthquakes $\beta = 1.6$
- Wait between trades of German bond futures $\beta \approx 0.95$
- Wait between Irish stock trades $\beta = 0.4$ (truncated)

# Fractional derivatives: An old idea gets new life

- Fractional derivatives $D^\beta f(x)$ for any $\beta > 0$ were invented by Leibniz (1695) soon after the more familiar integer derivatives.
- The Caputo fractional derivative of order $0 < \beta < 1$ defined by

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t d_s g(s) \frac{ds}{(t-s)^\beta} \tag{1}$$

  was invented to properly handle initial values (Caputo 1967).
- Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

## examples

- $$D_t^\beta(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$
- $$D_t^\beta(e^{\lambda t}) = \lambda^\beta e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$
- $$D_t^\beta(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

# Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = \partial_x^2 u(t,x); \quad u(0,x) = f(x) \tag{2}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(2) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(W(Q(t)))] = \int_0^\infty p(s,x) g_{Q(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(W(t))]$ and $Q(t) = \inf\{\tau > 0 : \ P(\tau) > t\}$, $P(t)$ is a stable subordinator with index $\beta$ and $\mathbb{E}(e^{-sP(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(W(Q(t)))^2 = \mathbb{E}(Q(t)) \approx t^\beta$.

Taking Fourier-Laplace transform of the Equation (2) gives

$$\begin{aligned} \bar{u}(s,k) &= \frac{s^{\beta-1}\hat{f}(k)}{s^{\beta}+k^2} \\ &= s^{\beta-1}\int_0^\infty \exp(-[s^\beta+k^2]l)\hat{f}(k)dl \end{aligned} \tag{3}$$

The next step is to invert this Fourier-Laplace transform using the fact that $Q(t)$ has density

$$f_{Q(t)}(s) = \frac{t}{\beta} g_\beta\left(\frac{t}{s^{1/\beta}}\right)s^{-1/\beta-1}, \text{ and } \int_0^\infty e^{-su} g_\beta(u) = e^{-s^\beta}.$$

In the case $\beta = 1/2$,

$$f_{Q(t)}(s) = \frac{2}{\sqrt{4\pi t}} e^{-s^2/4t} = f_{|W(t)|}(s)$$

This proof is due to Meerschaert, Benson, Scheffler and Baeumer (2002)

# Equivalence to Higher order PDE's

Let $\Delta f = \sum_{k=1}^{d} \partial_{x_k}^2 f(x)$, Laplacian of $f$.

- For any $m = 2, 3, 4, \ldots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} \Delta^j f(x) + \Delta^m u(t,x); \quad u(0,x) = f(x) \tag{4}$$

  and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = \Delta u(t,x); \quad u(0,x) = f(x), \tag{5}$$

  have the same unique solution given by

$$u(t,x) = \int_0^\infty p((t/s)^{1/m}, x) g_{1/m}(s)\, ds = E_x(f(W(Q(t))))$$

- Due to Baeumer, Meerschaert, and Nane TAMS(2009).

# Connections to iterated Brownian motions

- Orsingher and Benghin (2004) and (2008) show that for $\beta = 1/2^n$ the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{6}$$

is given by running

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|)$$

Where $W_j$'s are independent Brownian motions, i.e., $u(t,x) = E_x(f(I_{n+1}(t)))$ solves (6), and solves (4) for $m = 2^n$.

# Corollary

- We obtain the equivalence of one dimensional distributions in the case $Q(t)$ is the inverse stable subordinator of index $\beta = 1/2^n$

$$I_{n+1}(t) = W_1(|W_2(|W_3(|\cdots(W_{n+1}(t))\cdots|)|)|) \stackrel{(d)}{=} W_1(Q(t))$$

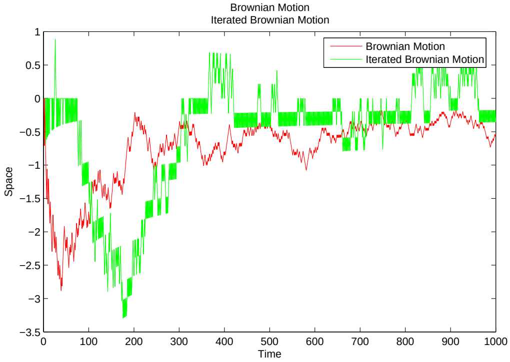


Figure: Simulations of iterated Brownian motions

# Heat equation in bounded domains

Heat equation in $D$ with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta u(t,x), \quad x \in D,\ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

When $D = (0, M)$, the heat equation can be solved by separation of variables: set $u(t,x) = \phi(x)T(t)$. Hence $\phi(x)$ satisfies

$$\partial_x^2 \phi(x) = -\lambda \phi(x), \quad x \in (0, M), \lambda > 0; \quad \phi(0) = 0, \quad \phi(M) = 0$$

and $T(t)$ satisfies $\partial_t T(t) = -\lambda T(t); \quad T(0) = 1.$

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}, T_n(t) = e^{-\lambda_n t}.$$

Same applies in any dimension $d \geq 1$.

Denote the eigenvalues and the eigenfunctions of $\Delta_D$ by $\{\lambda_n, \phi_n\}_{n=1}^{\infty}$, where $\phi_n \in C^\infty(D)$. The corresponding heat kernel is given by

$$p_D(t,x,y) = \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\phi_n(y).$$

The series converges absolutely and uniformly on $[t_0, \infty) \times D \times D$ for all $t_0 > 0$. In this case, the semigroup given by

$$\begin{aligned} T_D(t)f(x) &= E_x[f(W(t))I(t < \tau_D(X))] = \int_D p_D(t,x,y)f(y)dy \\ &= \sum_{n=1}^{\infty} e^{-\lambda_n t}\phi_n(x)\bar{f}(n) \end{aligned}$$

solves the Heat equation in $D$ with Dirichlet boundary conditions.

# Fractional diffusion in bounded domains

$$
\begin{aligned}
\partial_t^\beta u(t,x) &= \Delta u(t,x); \quad x \in D, \ t > 0 \\
u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.
\end{aligned} \tag{7}
$$

Separation of variables gives the unique (classical) solution as

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)M_\beta(-\lambda_n t^\beta) \\
&= E_x[f(W(Q(t)))I(\tau_D(W) > Q(t))] \\
&= E_x[f(W(Q(t)))I(\tau_D(W(Q)) > t)] \\
&= \frac{t}{\beta}\int_0^\infty T_D(l)f(x)g_\beta(tl^{-1/\beta})l^{-1/\beta-1}dl
\end{aligned}
$$

Joint work with Meerschaert and Vellaisamy, AOP (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

In this case, eigenfunctions and eigenvalues are

$$\phi_n(x) = \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M), \quad \lambda_n = \frac{\pi^2 n^2}{M^2}$$

The time fractional diffusion on $(0, M)$ has the solution

$$u(t,x) \;=\; \sum_{n=1}^{\infty} \bar{f}(n) \left(\frac{2}{M}\right)^{1/2} \sin(n\pi x/M) M_\beta(-\lambda_n t^\beta)$$

here

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^n}{\Gamma(1+\beta n)}$$

For $\beta = 1$, $M_1(-z) = e^{-z}$, and $u$ coincides with the solution of the heat equation on $(0, M)$.

# Sketch of Proof

- Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta_x u(t,x)dx = \int_D u(t,x)\Delta\phi_n(x)$$
$$= -\lambda_n \int_D u(t,x)\phi_n(x)dx = -\lambda_n \bar{u}(t,n)$$

Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\lambda_n \bar{u}(t,n). \tag{8}$$

- taking Laplace transforms on both sides of (8), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\lambda_n \hat{u}(s,n) \tag{9}$$

- Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\lambda_n}$.

# Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n) M_\beta(-\lambda_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\lambda Q(t)}) = \int_0^\infty e^{-\lambda l} g_{Q(t)}(l)\,dl = M_\beta(-\lambda t^\beta).$$

Inverting now the $\phi_n$-transform, we get an $L^2$-convergent solution of Equation (7) as (for each $t \geq 0$ )

$$u(t,x) = \sum^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\lambda_n t^\beta) \tag{10}$$

# Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$
\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\lambda_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) \int_0^\infty e^{-\lambda_n l} g_{Q(t)}(l)dl \\
&= \int_0^\infty \left( \sum_{n=1}^{\infty} \bar{f}(n) e^{-\lambda_n l} \phi_n(x) \right) g_{Q(t)}(l)dl \qquad (11) \\
&= \int_0^\infty T_D(l) f(x) g_{Q(t)}(l)dl \\
&= \int_0^\infty \mathbb{E}_x[f(B(l)) I(\tau_D > l)] g_{Q(t)}(l)dl \\
&= \mathbb{E}_x[f(B(Q(t))) I(\tau_D(B) > Q(t))]
\end{aligned}
$$

# IBM in bounded domains

The (classical) solution of

$$\begin{aligned} \partial_t u(t,x) &= \frac{\Delta f(x)}{\sqrt{\pi t}} + \Delta^2 u(t,x), \quad x \in D, \ t > 0; \qquad (12) \\ u(t,x) &= \Delta u(t,x) = 0, \ t \geq 0, \ x \in \partial D; \\ u(0,x) &= f(x), \ x \in D \end{aligned}$$

is given by (running $I_2(t) = W_1(|W_2(t)|)$, IBM)

$$\begin{aligned} u(t,x) &= E_x[f(I_2(t))I(\tau_D(W_1) > |W_2(t)|)] \\ &= 2\int_0^\infty T_D(l)f(x)h(t,l)dl, \qquad (13) \end{aligned}$$

where $T_D(l)$ is the heat semigroup in $D$, and $h(t,l)$ is the transition density of one-dimensional Brownian motion $\{W_2(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2$.

# Extensions

Uniformly elliptic operator of divergence form is defined on $C^2$ functions by

$$Lu = \sum_{i,j=1}^{d} \partial_{x_j} \left( a_{ij}(x)(\partial_{x_i} u) \right) \tag{14}$$

with $a_{ij}(x) = a_{ji}(x)$ and, for some $\lambda > 0$,

$$\lambda \sum_{i=1}^{n} y_i^2 \le \sum_{i,j=1}^{n} a_{ij}(x) y_i y_j \le \lambda^{-1} \sum_{i=1}^{n} y_i^2, \quad \forall y \in \mathbb{R}^d. \tag{15}$$

# New time operators

| Laplace symbol: $\psi(s)$ | inverse subordinator | time operator |
|---|---|---|
| $\int_0^\infty (1-e^{-sy})\nu(dy)$ | $Q_\psi(t)$ | $\psi(\partial_t) - \delta(0)\nu(t,\infty)$ |
| $s^\beta$ | $Q(t)$ | $\partial_t^\beta$, Caputo |
| $\int_0^1 s^\beta \Gamma(1-\beta)\mu(d\beta)$ | $Q_\mu(t)$ | $\int_0^1 \partial_t^\beta \Gamma(1-\beta)\mu(d\beta)$ |
| $(s+\lambda)^\beta - \lambda^\beta$ | $Q_\lambda(t)$ | $\partial_t^{\beta,\lambda}$ in (16) |

$$\begin{aligned} \partial_t^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t)g(t) - g(0)\phi_\lambda(t,\infty) \\ &= e^{-\lambda t}\frac{1}{\Gamma(1-\beta)} d_t\left[\int_0^t \frac{e^{\lambda s} g(s)\,ds}{(t-s)^\beta}\right] - \lambda^\beta g(t) \\ &\quad - \frac{g(0)}{\Gamma(1-\beta)}\int_t^\infty e^{-\lambda r}\beta r^{-\beta-1}\,dr. \end{aligned} \tag{16}$$

# New space operators

| Laplace exp.: $\psi(s)$ | subord. | process | Generator |
|---|---|---|---|
| $\int_0^\infty (1-e^{-sy})\nu(dy)$ | $D_\psi(t)$ | $W(D_\psi(t))$ | $\psi(-\Delta)$ |
| $s^\beta$ | $Q_\beta(t)$ | $W(Q_\beta(t))$ | $(-\Delta)^\beta$ |
| $(s+m^{1/\beta})^\beta - m$ | $T_\beta(t,m)$ | $W(T_\beta(t,m))$ | $(-\Delta + m^{1/\beta})^\beta - m$ |
| $\log(1+s^\beta)$ | $D_{\log}(t)$ | $W(D_{\log}(t))$ | $\log(1+(-\Delta)^\beta)$ |

$W(t)$, Brownian motion
$W(Q_\beta(t))$, symmetric stable process
$W(T_\beta(t,m))$, relativistic stable process
$W(D_{\log}(t))$, geometric stable process

# Other time-changes; Cauchy process

$X(t)$ a continuous Markov process with generator $\mathcal{A}$,
$Y(t)$ be a Cauchy process independent of $X(t)$. Then
$u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= -\frac{2\mathcal{A}f(x)}{\pi t} - \mathcal{A}^2 u(t,x), \ t > 0, \ x \in \mathbb{R}^d; \\ u(0,x) &= f(x) \ x \in \mathbb{R}^d. \end{aligned}$$

Due to Nane (2008).
Proof uses the fact that the density $p(t,s) = t/(\pi(s^2+t^2))$ of $Y(t)$ solves

$$(\partial_s^2 + \partial_t^2) p(t,s) = 0.$$

# Nonhomogeneous wave equation

This reduces to **nonhomogeneous wave equation** in the case $X$ is another Cauchy process independent of $Y$,
The generator of $X$ is $\mathcal{A} = -(-\Delta)^{1/2}$, fractional Laplacian.
$u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= \frac{2(-\Delta)^{1/2} f(x)}{\pi t} + \Delta u(t,x), \; t > 0, \; x \in \mathbb{R}^d; \\ u(0,x) &= f(x), \; x \in \mathbb{R}^d \end{aligned}$$

This is one of the most interesting PDE connections of these iterated processes.

# Further research

- Work in progress for the **subordinated Brownian motions**, e.g. symmetric stable process as the outer process. The corresponding space operators are $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$
- Extension to Neumann boundary conditions...
- Fractal properties of $W(Q(t))$ and other subordinate processes
- Work in progress for the Subordinated Brownian motions, e.g. symmetric stable process as the outer process....
- Applications-interdisciplinary research

# Thank You!