Fractional Cauchy Problems for time-changed Processes

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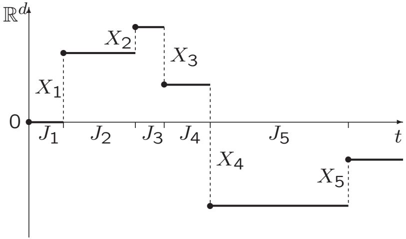
Continuous time random walks



The CTRW is a random walk with jumps X_n separated by random waiting times J_n. The random vectors (X_n, J_n) are i.i.d.

Waiting time process

J_n's are nonnegative iid.
$T_n = J_1 + J_2 + \cdots + J_n$ is the time of the nth jump.
$N(t) = \max\{n \geq 0 : \ T_n \leq t\}$ is the number of jumps by time $t > 0$.
Suppose $P(J_n > t) \approx Ct^{-\beta}$ for $0 < \beta < 1$.
Scaling limit

$$c^{-1/\beta} T_{[ct]} \implies D(t)$$

is a β-stable subordinator.
Since $\{T_n \leq t\} = \{N(t) \geq n\}$

$$c^{-\beta} N(ct) \implies E(t) = \inf\{u > 0 : \quad D(u) > t\}.$$

The self-similar limit $E(ct) \overset{d}{=} c^{\beta} E(t)$ is non-Markovian.

Continuous time random walks (CTRW)

$S(n) = X_1 + \cdots + X_n$ particle location after n jumps
has scaling limit $c^{-1/2}S([ct]) \implies B(t)$ a Brownian motion.
Number of jumps has scaling limit $c^{-\beta}N(ct) \implies E(t)$.
CTRW is a random walk subordinated to (a renewal process) $N(t)$

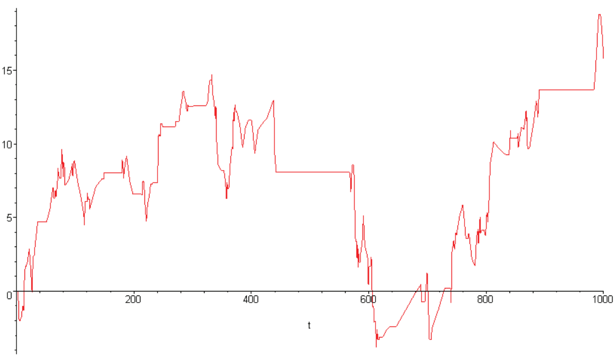
$$S(N(t)) = X_1 + X_2 + \cdots + X_{N(t)}.$$

$S(N(t))$ is particle location at time $t > 0$.
CTRW scaling limit is a subordinated process:

$$\begin{aligned} c^{-\beta/2}S(N(ct)) &= (c^{\beta})^{-1/2}S(c^{\beta}\cdot c^{-\beta}N(ct)) \\ &\approx (c^{\beta})^{-1/2}S(c^{\beta}E(t)) \implies B(E(t)). \end{aligned}$$

The self-similar limit $B(E(ct)) \overset{d}{=} c^{\beta/2}B(E(t))$ is non-Markovian.

Scaling limit: Subordinated motion



Limit retains long waiting times.

Power law waiting times

Wait between solar flares $1 < \beta < 2$

Wait between raindrops $\beta = 0.68$

Wait between money transactions $\beta = 0.6$

Wait between emails $\beta \approx 1.0$

Wait between doctor visits $\beta \approx 1.4$

Wait between earthquakes $\beta = 1.6$

Wait between trades of German bond futures $\beta \approx 0.95$

Wait between Irish stock trades $\beta = 0.4$ (truncated)

CTRW with serial dependence

Particle jumps $X_n = \sum_{j=0}^{\infty} c_j Z_{n-j}$ with Z_n IID.

Short range dependence: $\sum_{n=1}^{\infty} |\mathbb{E}(X_n X_0)| < \infty \implies$ the usual limit and PDE.

Long range dependence: If Z_n has light tails, subordinated fractional Brownian motion limit $B_H(E(t))$. Hahn, Kobayashi and Umarov (2010) established a governing equation.

For heavy tails, subordinated linear fractional stable motion $L_{\alpha,H}(E(t))$.

Open problems: Governing equations, dependent waiting times.

Due to Meerschaert, Nane and Xiao (2009).

Fractional time derivative: Two approaches

Riemann-Liouville fractional derivative of order $0 < \beta < 1$;

$$\mathbb{D}_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \frac{\partial}{\partial t} \left[\int_0^t g(s) \frac{ds}{(t-s)^\beta} \right]$$

with Laplace transform $s^\beta \tilde{g}(s)$, $\tilde{g}(s) = \int_0^\infty e^{-st} g(t) dt$ denotes the usual Laplace transform of g.

Caputo fractional derivative of order $0 < \beta < 1$;

$$D_t^\beta g(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{dg(s)}{ds} \frac{ds}{(t-s)^\beta} \tag{1}$$

was invented to properly handle initial values (Caputo 1967). Laplace transform of $D_t^\beta g(t)$ is $s^\beta \tilde{g}(s) - s^{\beta-1} g(0)$ incorporates the initial value in the same way as the first derivative.

examples

$$D_t^{\beta}(t^p) = \frac{\Gamma(1+p)}{\Gamma(p+1-\beta)} t^{p-\beta}$$

$$D_t^{\beta}(e^{\lambda t}) = \lambda^{\beta} e^{\lambda t} - \frac{t^{-\beta}}{\Gamma(1-\beta)}?$$

$$D_t^{\beta}(\sin t) = \sin(t + \frac{\pi\beta}{2})$$

Diffusion

Let L_x be the generator of some continuous Markov process $X(t)$. Then $p(t,x) = \mathbb{E}_x[f(X(t))]$ is the unique solution of the heat-type Cauchy problem

$$\partial_t p(t,x) = L_x p(t,x),\ t>0, x\in\mathbb{R}^d;\quad p(0,x) = f(x),\ x\in\mathbb{R}^d$$

Examples:
X: Brownian motion then $L_x = \Delta_x$, BM is a stochastic solution of the heat equation
X: Symmetric α-stable process then $L_x = -(-\Delta)^{\alpha/2}$

Time-fractional model for Anamolous sub-diffusion

Let $0 < \beta < 1$. Nigmatullin (1986) gave a physical derivation of fractional diffusion

$$\partial_t^\beta u(t,x) = L_x u(t,x); \quad u(0,x) = f(x) \tag{2}$$

Zaslavsky (1994) used this to model Hamiltonian chaos.
(2) has the unique solution

$$u(t,x) = \mathbb{E}_x[f(X(E(t)))] = \int_0^\infty p(s,x) g_{E(t)}(s) ds$$

where $p(t,x) = \mathbb{E}_x[f(X(t))]$ and $E(t) = \inf\{\tau > 0 : \ D_\tau > t\}$, $D(t)$ is a stable subordinator with index β and $\mathbb{E}(e^{-sD(t)}) = e^{-ts^\beta}$ (Baeumer and Meerschaert, 2002).
$\mathbb{E}_x(B(E(t))) = x\mathbb{E}(E(t)^{1/2}) \approx xt^{\beta/2}$.

Equivalence to higher order PDE's

For any $m = 2, 3, 4, \dots$ both the Cauchy problem

$$\partial_t u(t,x) = \sum_{j=1}^{m-1} \frac{t^{j/m-1}}{\Gamma(j/m)} L_x^j f(x) + L_x^m u(t,x); \quad u(0,x) = f(x) \tag{3}$$

and the fractional Cauchy problem:

$$\partial_t^{1/m} u(t,x) = L_x u(t,x); \quad u(0,x) = f(x), \tag{4}$$

have the same unique solution given by

$$u(t,x) = \mathbb{E}_x[f(X(E^{1/m}(t)))] = \int_0^\infty p(s,x) g_{E^{1/m}(t)}(s)\, ds$$

Due to Allouba and Zheng (2001), Baeumer, Meerschaert, and Nane (2007), Keyantuo and Lizama (2009), Li et al. (2010).

Connections to iterated Brownian motions

Orsingher and Beghin (2004, 2008) show that the solution to

$$\partial_t^{1/2^n} u(t,x) = \Delta_x u(t,x); \quad u(0,x) = f(x), \tag{5}$$

is given by running

$$I_{n+1}(t) = B_1(|B_2(|B_3(|\cdots(B_{n+1}(t))\cdots|)|)|)$$

Where B_j's are independent Brownian motions, i.e.,
$u(t,x) = \mathbb{E}_x(f(I_{n+1}(t)))$ solves (5), and solves (3) for $m = 2^n$.

Heat equation in bounded domains

Denote the eigenvalues and the eigenfunctions of Δ on a bounded domain D with Dirichlet boundary conditions by $\{\mu_n, \phi_n\}_{n=1}^{\infty}$;

$$\Delta\phi_n(x) = -\mu_n\phi_n(x), \quad x \in D; \phi_n(x) = 0, \quad x \in \partial D.$$

$\tau_D(X) = \inf\{t \geq 0 : \ X(t) \notin D\}$ is the first exit time of a process X, and let $\bar{f}(n) = \int_D f(x)\phi_n(x)dx$. The semigroup given by

$$T_D(t)f(x) \quad = \quad \mathbb{E}_x[f(B(t))I(\tau_D(B) > t)] = \sum_{n=1}^{\infty} e^{-\mu_n t}\phi_n(x)\bar{f}(n)$$

solves the heat equation in D with Dirichlet boundary conditions:

$$\begin{aligned} \partial_t u(t,x) &= \Delta_x u(t,x), \quad x \in D, \ t > 0, \\ u(t,x) &= 0, \quad x \in \partial D; \quad u(0,x) = f(x), \quad x \in D. \end{aligned}$$

Fractional diffusion in bounded domains

$$\begin{aligned}\partial_t^\beta u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.\end{aligned} \tag{6}$$

has the unique (classical) solution

$$\begin{aligned}u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\mu_n t^\beta) = \int_0^\infty T_D(l)f(x)g_{E(t)}(l)dl \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\ &= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)]\end{aligned}$$

Joint work with Meerschaert and Vellaisamy (2009).

Analytic solution in intervals $(0, M) \subset \mathbb{R}$ was obtained by Agrawal (2002).

Sketch of Proof

Use Green's second identity and Dirichlet b.c. to write

$$\int_D \phi_n(x)\Delta_x u(t,x)dx = \int_D u(t,x)\Delta\phi_n(x)$$
$$= -\mu_n \int_D u(t,x)\phi_n(x)dx = -\mu_n \bar{u}(t,n)$$

Apply to both sides of the fractional Cauchy problem to get

$$\partial_t^\beta \bar{u}(t,n) = -\mu_n \bar{u}(t,n). \tag{7}$$

taking Laplace transforms on both sides of (7), we get

$$s^\beta \hat{u}(s,n) - s^{\beta-1}\bar{u}(0,n) = -\mu_n \hat{u}(s,n) \tag{8}$$

Collecting the like terms leads to $\hat{u}(s,n) = \frac{\bar{f}(n)s^{\beta-1}}{s^\beta+\mu_n}$.

Sketch of Proof (page2)

By inverting the above Laplace transform, we obtain

$$\bar{u}(t,n) = \bar{f}(n) M_\beta(-\mu_n t^\beta)$$

in terms of the Mittag-Leffler function defined by

$$M_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+\beta k)}.$$

Compute the Laplace transform of the hitting time density

$$\mathbb{E}(e^{-\mu E(t)}) = \int_0^\infty e^{-\mu l} g_{E(t)}(l)\,dl = M_\beta(-\mu t^\beta).$$

Inverting now the ϕ_n-transform, we get an L^2-convergent solution of Equation (6) as (for each $t \geq 0$)

$$u(t,x) = \sum_{n=1}^{\infty} \bar{f}(n) \phi_n(x) M_\beta(-\mu_n t^\beta) \tag{9}$$

Sketch of Proof (page3)

To get the probabilistic form of the solution we proceed as

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) M_\beta(-\mu_n t^\beta) \\
&= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x) \int_0^\infty e^{-\mu_n l} g_{E(t)}(l)\,dl \\
&= \int_0^\infty \left(\sum_{n=1}^{\infty} \bar{f}(n) e^{-\mu_n l} \phi_n(x) \right) g_{E(t)}(l)\,dl \\
&= \int_0^\infty T_D(l) f(x) g_{E(t)}(l)\,dl \\
&= \int_0^\infty \mathbb{E}_x[f(B(l)) I(\tau_D > l)] g_{E(t)}(l)\,dl \\
&= \mathbb{E}_x[f(B(E(t))) I(\tau_D(B) > E(t))]
\end{aligned} \tag{10}$$

IBM in bounded domains

The (classical) solution of

$$\begin{aligned}
\partial_t u(t,x) &= \sum_{j=1}^{2^n-1} \frac{t^{j/2^n-1}}{\Gamma(j/m)} \Delta_x^j f(x) + \Delta_x^{2^n} u(t,x), \quad x \in D, \ t > 0; \\
u(t,x) &= \Delta_x^l u(t,x) = 0, \ t \geq 0, \ x \in \partial D, \ l = 1, \cdots 2^n - 1; \\
u(0,x) &= f(x), \ x \in D
\end{aligned}$$

is given by (running $I_{n+1}(t) = B_1(|B_2(\cdots|B_{n+1}(t)|)|) = B_1(|I_n(t)|))$

$$\begin{aligned}
u(t,x) &= E_x[f(I_{n+1}(t)) I(\tau_D(B_1) > |I_n(t)|)] \\
&= 2\int_0^\infty T_D(l) f(x) h(t,l) dl, \qquad (11)
\end{aligned}$$

where $h(t,l)$ is the transition density of $\{I_n(t)\}$.
Proof: equivalence with fractional Cauchy problem for $\beta = 1/2^n$.

Corollary

$$||u(t,\cdot)||_{L^2} \sim CE_\beta(-\mu_1 t^\beta) \sim \frac{C}{\mu_1 t^\beta}$$

In the Heat-equation case, since $\beta = 1$ we have $E_\beta(-\mu_1 t^\beta) = e^{-\mu_1 t}$ so

$$||u(t,\cdot)||_{L^2} \sim CE_1(-\mu_1 t) = Ce^{-\mu_1 t}$$

Stochastic model for ultraslow diffusion

Let $supp\nu \subset (0,1)$ be a finite measure.
$W(t)$, increasing Lévy process with laplace transform

$$\mathbb{E}[e^{-sW(t)}] = e^{-t\left[\int_0^\infty (e^{-sx}-1)\phi_W(dx)\right]} = e^{-t\left[\int_0^1 s^\beta \Gamma(1-\beta)\nu(d\beta)\right]}.$$

The Lévy measure is

$$\phi_W(t,\infty) = \int_0^1 t^{-\beta}\nu(d\beta), \tag{12}$$

Let $E(t) = \inf\{\tau \geq 0 : \ W(\tau) \geq t\}$ be the inverse process.
$B(E(t))$ is a stochastic solution of

$$\mathbb{D}^{(\nu)}u(t,x) := \int_0^1 \partial_t^\beta u(t,x)\Gamma(1-\beta)\nu(d\beta) = \Delta_x u(t,x)$$

For $\nu(d\beta) = \beta^{\alpha-1}d\beta, \alpha > 0$,
$\mathbb{E}(B(E_W(t))^2) = \mathbb{E}(E_W(t)) \sim C(\alpha)(\log t)^\alpha \text{ as } t \to \infty.$

Ultraslow diffusion in bounded domains

$$\begin{aligned}
\mathbb{D}^{(\nu)}u(t,x) &= \Delta_x u(t,x); \quad x \in D, \ t > 0 \\
u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.
\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned}
u(t,x) &= \sum_{n=1}^{\infty} \bar{f}(n)\phi_n(x)h(t,\mu_n) = \int_0^{\infty} T_D(l)f(x)g_{E(t)}(l)dl \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B) > E(t))] \\
&= \mathbb{E}_x[f(B(E(t)))I(\tau_D(B(E)) > t)]
\end{aligned}$$

where $h(t,\mu_n) = \mathbb{E}(e^{-\mu_n E(t)})$.

Joint work with Meerschaert and Vellaisamy (2009).

Eigenvalue problem for distributed order time derivative

$h(t,\mu) = \mathbb{E}(e^{-\mu E(t)})$ is the solution of

$$\mathbb{D}^{(\nu)} h(t,\mu) = -\mu h(t,\mu); \quad h(0,\mu) = 1. \tag{13}$$

In the case $\nu(d\beta) = p(\beta)d\beta$; by inverse Laplace transform it has the representation

$$h(t,\mu) = \frac{\mu}{\pi} \int_0^\infty r^{-1} e^{-tr} \Phi(r,1) dr \tag{14}$$

where for $U(r) = \int_0^1 r^\beta \sin(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta$

$$\Phi(r,1) = \frac{U(r)}{[\int_0^1 r^\beta \cos(\beta\pi)\Gamma(1-\beta)p(\beta)d\beta + \mu]^2 + [U(r)]^2}.$$

Due to Kochubei (2008).

Tempered fractional model

$D_\lambda(t)$, increasing Lévy process with laplace transform

$$\mathbb{E}[e^{-sD_\lambda(t)}] = e^{-t\psi_\lambda(s)} = e^{-t\left[\int_0^\infty (e^{-sx}-1)\phi_{D_\lambda}(dx)\right]} = e^{-t\left[(s+\lambda)^\beta - \lambda^\beta\right]}.$$

The Lévy measure is

$$\phi_{D_\lambda}(t,\infty) = \frac{1}{\Gamma(1-\beta)} \int_t^\infty e^{-\lambda r} \beta r^{-\beta-1}\, dr, \tag{15}$$

Let $E_\lambda(t) = \inf\{\tau \geq 0 : \ D_\lambda(\tau) \geq t\}$ be the inverse process. Define the caputo tempered fractional time derivative by

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} g(t) &= \psi_\lambda(\partial_t)g(t) - g(0)\phi_{D_\lambda}(t,\infty) \\ &= e^{-\lambda t} \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_0^t \frac{e^{\lambda s} g(s)\, ds}{(t-s)^\beta} - \lambda^\beta g(t) \\ &\quad - g(0)\phi_{D_\lambda}(t,\infty)\end{aligned} \tag{16}$$

Tempered fractional cauchy problem

Then $\mathbb{E}[f(B(E_\lambda(t)))]$ is a stochastic solution of

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} u(t,x) = \Delta_x u(t,x);\ u(t,0) = f(x).$$

Due to Meerschaert and Sheffler (2008).

$$\mathbb{E}_x(B(E_\lambda(t)))^2 \approx \begin{cases} t^\beta/\Gamma(1+\beta), & t << 1 \\ t/\beta, & t >> 1. \end{cases}$$

$B(E_\lambda(t))$ occupies an intermediate place between subdiffusion and diffusion (Stanislavsky et al., 2008)

Tempered fractional cauchy problem in bounded domains

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} u(t,x) &= \Delta_x u(t,x), \quad x \in D, \ t > 0; \\ u(t,x) &= 0, \ x \in \partial D, \ t > 0; \quad u(0,x) = f(x), \ x \in D.\end{aligned}$$

has the unique (classical) solution

$$\begin{aligned} u(t,x) &= \mathbb{E}_x(f(B(E_\lambda(t)))I(\tau_{D_\lambda}(B) > E_\lambda(t)) \\ &= \int_0^\infty T_D(l)f(x)g_\lambda(t,l)dl \\ &= \sum_{n=1}^\infty \bar{f}(n)\phi_n(x)h_\lambda(t,\mu_n), \end{aligned}$$

where $h_\lambda(t,\mu) = \mathbb{E}(e^{-\mu E_\lambda(t)}) = \int_0^\infty e^{-\mu x} g_\lambda(t,x)\,dx$ is the Laplace transform of $E_\lambda(t)$ (Meerschaert, Nane and Vellaisamy 2010).

Eigenvalue problem for tempered fractional time derivative

$h_\lambda(t,\mu) = \mathbb{E}(e^{-\mu E_\lambda(t)})$ is the unique solution of

$$\left(\frac{\partial}{\partial t}\right)^{\beta,\lambda} h_\lambda(t,\mu) = -\mu h_\lambda(t,\mu); \quad h_\lambda(0,\mu) = 1. \tag{17}$$

For any $\mu, \lambda > 0$, $\mu \neq \lambda^\beta$, the function $h_\lambda(t,\mu)$ has the representation

$$h_\lambda(t,\mu) = \frac{\mu}{\pi} \int_0^\infty (r+\lambda)^{-1} e^{-t(r+\lambda)} \Phi(r,1)\, dr, \tag{18}$$

where

$$\Phi(r,1) = \frac{r^\beta \sin(\beta\pi)}{r^{2\beta} \sin^2(\beta\pi) + (\mu - \lambda^\beta + r^\beta \cos(\beta\pi))^2}$$

Due to Meerschaert, Nane and Vellaisamy (2010).

Other Subordinators; Cauchy process

$X(t)$ a continuous Markov process with generator $\mathcal{A}$,
$Y(t)$ be a Cauchy process independent of $X(t)$. Then
$u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= -\frac{2\mathcal{A}f(x)}{\pi t} - \mathcal{A}^2 u(t,x), \ t>0, \quad x \in \mathbb{R}^d; \\ u(0,x) &= f(x) \ x \in \mathbb{R}^d. \end{aligned}$$

Due to Nane (2008).
Proof uses the fact that the density $p(t,s)$ of $Y(t)$ solves

$$(\partial_s^2 + \partial_t^2) p(t,s) = 0.$$

Nonhomogeneous wave equation

This reduces to **nonhomogeneous wave equation** in the case X is another Cauchy process independent of Y,
The generator of X is $\mathcal{A} = -(-\Delta)^{1/2}$, fractional Laplacian.
$u(t,x) = \mathbb{E}_x[f(X(|Y(t)|))]$ is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) \quad &= \quad \frac{2(-\Delta)^{1/2} f(x)}{\pi t} \; + \; \Delta u(t,x), \; t > 0, \;\; x \in \mathbb{R}^d; \\ u(0,x) \quad &= f(x), \; x \in \mathbb{R}^d \end{aligned}$$

This is one of the most interesting PDE connections of these iterated processes.

Bounded domains

Let D be a bounded domain. Then

$$\begin{aligned} u(t,x) &= \mathbb{E}_x[f(B(|Y(t)|)I(\tau_D(B) > |Y(t)|)] \\ &= 2\int_0^\infty \left(\sum_{n=1}^{\infty} e^{-\mu_n s}\bar{f}(n)\phi_n(x) \right) \frac{t}{\pi(t^2+s^2)} ds \end{aligned}$$

is a solution of

$$\begin{aligned} \partial_t^2 u(t,x) &= -\frac{2\Delta f(x)}{\pi t} - \Delta^2 u(t,x), \quad t>0, \ \ x \in D \qquad (19) \\ u(0,x) &= f(x), \qquad x \in D, \\ u(t,x) &= \Delta u(t,x) = 0, \ \ x \in \partial D, \ t>0. \end{aligned}$$

Due to Nane (2010).

Stable subordinators: $\alpha \neq 1$

$\alpha \in (0,2)$ be rational $\alpha = l/m$, where l and m are relatively prime.
$Y(t)$; a symmetric α-stable process
Then $u(t,x) = \mathbb{E}_x[f(B(|Y(t)|)]$ is a solution of

$$\begin{aligned} (-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) &= -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p^{\alpha}(t,s)\Big|_{s=0}\right)\Delta^{2i-1}f(x) \\ &\quad - \Delta^{2l}u(t,x), \qquad t>0, \quad x \in \mathbb{R}^d \\ u(0,x) &= f(x), \qquad x \in \mathbb{R}^d. \end{aligned}$$

Where $p^{\alpha}(t,s)$ is the transition density of symmetric α-stable process $Y(t)$. The proof uses the fact that

$$\partial_t^{2m}p^{\alpha}(t,x) = (-\partial_x^2)^l p^{\alpha}(t,x)$$

Bounded domains

$\alpha \in (0,2)$ a rational $\alpha = l/m$, where l and m are relatively prime.
D a bounded domain.
Then $u(t,x) = \mathbb{E}_x[f(B(|Y(t)|)I(\tau_D(B) > |Y(t)|)]$ is a classical solution of

$$\begin{aligned}
(-1)^{l+1}\frac{\partial^{2m}}{\partial t^{2m}}u(t,x) &= -2\sum_{i=1}^{l}\left(\frac{\partial^{2l-2i}}{\partial s^{2l-2i}}p^{\alpha}(t,s)\Big|_{s=0}\right)\Delta^{2i-1}f(x) \qquad (20\\
&\quad - \Delta^{2l}u(t,x), \qquad t>0, \quad x\in D\\
u(0,x) &= f(x), \qquad x\in D.\\
u(t,x) &= \Delta^{j}u(t,x) = 0, \quad x\in\partial D, \; t>0, \; j = 1,\cdots,2l-1
\end{aligned}$$

Further research

Work in progress for the **subordinated Brownian motions**, e.g. symmetric stable process as the outer process. The corresponding space operators are $(-\Delta)^{\alpha/2}$ for $0 < \alpha \leq 2$

Extension to Neumann boundary conditions...

Fractal properties of $B(E(t))$ and other subordinate processes

Applications-interdisciplinary research

Further research:a question?

D'ovidio and Orsingher (2010) established the fact that the density of $B^H(|B(t)|) \overset{1-d}{=} B^H(E(t))$ for $\beta = 1/2$

$$q(t,x) = 2\int_0^\infty \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \frac{e^{-\frac{s^2}{2t}}}{\sqrt{2\pi t}} ds$$

is a solution of the first order PDE

$$t\frac{\partial q(t,x)}{\partial t} = -\frac{H}{2}\frac{\partial}{\partial x}(xq(t,x)), \quad t > 0,\, x \in \mathbb{R}. \tag{21}$$

How about $B^H(|B_1(|B_2(\cdots|(B_n(t)|)|)|)|) \overset{1-d}{=} B^H(E(t))$ for $\beta = 1/2^n$?

By induction, the density of $B^H(|B_1(|B_2(\cdots|(B_n(t)|)|)|)|) \overset{1-d}{=} B^H(E(t))$ is a solution to the first-order pde:

$$t\frac{\partial q(t,x)}{\partial t} = -\frac{H}{2^n}\frac{\partial}{\partial x}(xq(t,x)), \quad t>0,\ x\in\mathbb{R}. \tag{22}$$

The first order PDE

$$t\frac{\partial u(t,x)}{\partial t} = -H_1H_2\frac{\partial}{\partial x}(xu(t,x)), \quad t>0,\ x\in\mathbb{R}. \tag{23}$$

has a general solution of the form

$$u(t,x) = \frac{1}{x}f\left(\frac{x}{t^{H_1H_2}}\right), \quad x\in\mathbb{R}\setminus\{0\},\ t>0$$

with $H_1, H_2 \in (0,1)$ and $f \in C^1(\mathbb{R})$.

An approach for higher order PDES

To get the Higher order PDEs for $B^H(|B_1(|B_2(\cdots|(B_n(t)|)|)|)|) \overset{1-d}{=} B^H(E(t))$. Let $\beta = \frac{1}{m}$ then

$$\begin{aligned} \frac{\partial f_{E(t)}(s)}{\partial t} &= (-1)^m \frac{\partial^m f_{E(t)}(s)}{\partial s^m}, \quad s, t > 0; \\ \frac{\partial^k f_{E(t)}(0)}{\partial s^k} &= \frac{(-1)^k t^{-(k+1)/m}}{\Gamma(1 - \frac{(k+1)}{m})}, \quad t > 0, k = 0, 1, 2, \cdots, m-1; \\ \lim_{s\to\infty} \frac{\partial^k f_{E(t)}(s)}{\partial s^k} &= 0, \quad t > 0, k = 0, 1, 2, \cdots, (m-1). \end{aligned} \tag{24}$$

By Keyantuo and Lizama (2010).

In the case $0 < H < 1$ and $\alpha = 1$ the density function

$$q(t,x) = 2\int_0^\infty f^H(s,x)p_t(s)ds = 2\int_0^\infty \frac{e^{-\frac{|x|^2}{2s^{2H}}}}{(2\pi s^{2H})^{d/2}} \frac{t}{\pi(t^2+s^2)}ds$$

of $W(Y(t))$ solves the PDE

$$\begin{aligned}\frac{\partial^2 q(t,x)}{\partial t^2} = &-\frac{2HI_{(0,1/2]}(H)}{\pi t}\Delta\delta(x) - H(2H-1)\Delta G_{(2H-2),t}q(t,x) \\ &- H^2\Delta^2 G_{(4H-2),t}q(t,x), \qquad x \in \mathbb{R}^d, t > 0,\end{aligned} \tag{25}$$

where

$$G_{\gamma,t}q(t,x) = 2\int_0^\infty s^\gamma p_t(s) f^H(s,x)ds, \quad \gamma \neq 0,$$

and $G_{0,t}$ is the identity operator. Due to Nane, Xiao and Wu (2010). Properties of $G_{\gamma,t}$?

Beghin, Orsingher and Sakhno (2010) has established that the density $W(Y(t))$ in the case $d = 1$, $0 < H < 1, \alpha = 1$

$$q(t,x) = 2\int_0^\infty \frac{e^{-\frac{x^2}{2s^{2H}}}}{\sqrt{2\pi s^{2H}}} \frac{t}{\pi(t^2+s^2)} ds$$

solves

$$\begin{aligned} \frac{\partial^2}{\partial t^2} q(t,x) &= -\frac{1}{t^2}\left[H(H-1)\frac{\partial}{\partial x}x - H^2\frac{\partial^2}{\partial x^2}x^2\right] q(t,x) \\ &- \frac{2HI_{(0,1/2]}(H)}{\pi t}\frac{\partial^2 \delta(x)}{\partial x^2}. \end{aligned} \tag{26}$$

Thank You!