Brownian Motion — Chapter 8:

"Recall"

Caratheodory thm

A (countably additive) probability P on an algebra (= field) $\mathcal{F}_0$ has a unique extension to a probability on $\sigma(\mathcal{F}_0) = \mathcal{F}$.

proof: For $A \subseteq \Omega$

outer measure $$m^*(A) = \inf_{\substack{B \in \mathcal{F}_0 \\ B \supseteq A}} P(B)$$

inner measure $$m_*(A) = 1 - m^*(A^c)$$

then
$$\mathcal{M} = \text{"measurable sets"} = \{A \subseteq \Omega : m^*(A) = m_*(A)\}.$$

is a σ-field (containing $\mathcal{F}_0$) and m^* is a p.m on $\mathcal{M}$.

Remark: Let F be a c.d.f. on $\mathbb{R}$ (or $\mathbb{R}^k$)

once we have shown that the assignment of ~~probabilit~~ probabilities on the algebra of finite unions of intervals (or "rectangles") is countably additive, the Caratheodory thm says we can extend the probability assignment (uniquely) to the σ-algebra of Borel sets.

<u>Special case</u>: Length (= Leb. measure) on $[0,1]$, area on $[0,1]^2$, Volume on $[0,1]^3$ ... etc.

<u>Definition</u>: Let T be an arbitrary set. A stochastic process with index set T is a collection of random variables $\{X_t : t \in T\}$ on a probability space $(\Omega, \mathcal{F}, P)$.

<u>Examples</u>: (i) $X_1, X_2 \ldots$ iid $\sim F$.

(ii) $S_0, S_1, S_2, \ldots$ $S_n = \sum_{i=1}^{n} X_i$, X_i's from (i)

(iii) Poisson process $\{N(t), t \in [0,\infty)\}$

with

$N(t) \sim \text{Poisson}(\lambda t)$, $\lambda =$ "rate"

stationary, independent increments.

Non-decreasing paths., $N(t,\omega) \nearrow$ in t $\forall \omega \in \Omega$.

(iv) Standard Brownian motion $\{W(t), t \geq 0\}$

$W(0) = 0$

$W(t_2) - W(t_1) \sim N(\text{mean} = 0, \text{var} = |t_2 - t_1|)$

independent increments

continuous paths: $\forall \omega \in \Omega$ $W(t,\omega)$ cont. in t.

(v) Standard Gamma process $\{X_t : t \geq 0\}$

$X_0 = 0$

$X_{t_2} - X_{t_1} \sim \Gamma(t_2 - t_1, 1)$ $t_2 > t_1 \geq 0$

Independent increments
Non-decreasing paths

Recall ; $\Gamma(1,1) = Exp(1)$

$$\underbrace{\Gamma(a,1) + \Gamma(b,1)}_{\text{indep}} \sim \Gamma(a+b, 1)$$

<u>Finite dimensional distributions</u> :

Def: If $\{X_t : t \in T\}$ is a stochastic process, then for each vector $(t_1, t_2, \ldots t_k)$ with entries from T $(X_{t_1}, \ldots X_{t_k})$ is a random vector in $\mathbb{R}^k$ with a distribution $\mu_{t_1, \ldots t_k}$. The p.m's $\mu_{t_1, \ldots t_k}$ are called the finite dimensional distribution of the stochastic process.

If $H_1, \ldots H_k$ are real Borel sets (or just intervals), then the fidi's obviously satisfy the consistency conditions ;

(i) $\mu_{t_1, \ldots t_k}(H_1 \times \ldots \times H_k)$

$= \mu_{t_{\pi(1)}, \ldots t_{\pi(k)}}(H_{\pi(1)} \times \ldots \times H_{\pi(k)})$

for any permutation π of $(1, 2, \ldots, k)$

(ii) $\mu_{t_1, \ldots t_{k-1}}(H_1 \times \ldots \times H_{k-1})$

$= \mu_{t_1, \ldots t_{k-1}, t_k}(H_1 \times H_2 \times \ldots \times H_{k-1} \times \mathbb{R})$

preview: If $\{\mu_{t_1,\dots t_k},\ \{t_1,\dots t_k\} \subseteq T\}$ is a consistent system of fidi's, then there exists a stochastic process with these fidi's.

Remark: if $T \subseteq \mathbb{R}$ (or is otherwise ordered) we can just consider fidis

$\mu_{t_1,\dots t_k}$ with $t_1 < t_2 < \dots < t_k$

in which case we do not need condition (i) above.

Remark: Check that the fidi specifications in the examples are consistent.

Definition: Let $\overline{\mathbb{R}}^T$ = all functions from T to $\overline{\mathbb{R}} = [-\infty, \infty]$ (and $\mathbb{R}^T$ = all functions from T to $\mathbb{R} = (-\infty, \infty)$.) Thus if $x \in \overline{\mathbb{R}}^T$, then $x : T \to \overline{\mathbb{R}}$. For each $t \in T$, define the coordinate function $Z_t : \overline{\mathbb{R}}^T \to \overline{\mathbb{R}}$ by

$$Z_t(x) = x(t), \quad \forall x \in \overline{\mathbb{R}}^T.$$

Definition: $\overline{\mathcal{R}}^T$ = σ-field generated by coordinate functions

= smallest σ-field such that coordinate functions are measurable.

Note: If $\{X_t, t \in T\}$ is a stochastic process on $(\Omega, \mathcal{F}, P)$, then for each $\omega \in \Omega$ $X(t, \omega) \in \overline{\mathbb{R}}^T$. This gives us a map $X : \Omega \longrightarrow \overline{\mathbb{R}}^T$. It is easy to see that this map is $\mathcal{F} / \overline{\mathcal{R}}^T$ measurable.

Thus we get an induced probability measure P' on $(\overline{\mathbb{R}}^T, \overline{\mathcal{R}}^T)$ which should perhaps be called the distribution of the process.

$A \in \overline{\mathcal{R}}$
$P'(A) = P(X \in A) = P(X^{-1}(A))$

VFY: The coordinate functions $\{Z_t, t \in T\}$ form a stochastic process on $(\overline{\mathbb{R}}^T, \overline{\mathcal{R}}^T, P')$ with the same fidi's as the original $\{X_t, t \in T\}$

Thus if we wish to get a stochastic process with specifed fidi's the we will be able to get it on $(\overline{\mathbb{R}}^T, \overline{\mathcal{R}}^T)$ if we can get it at all.

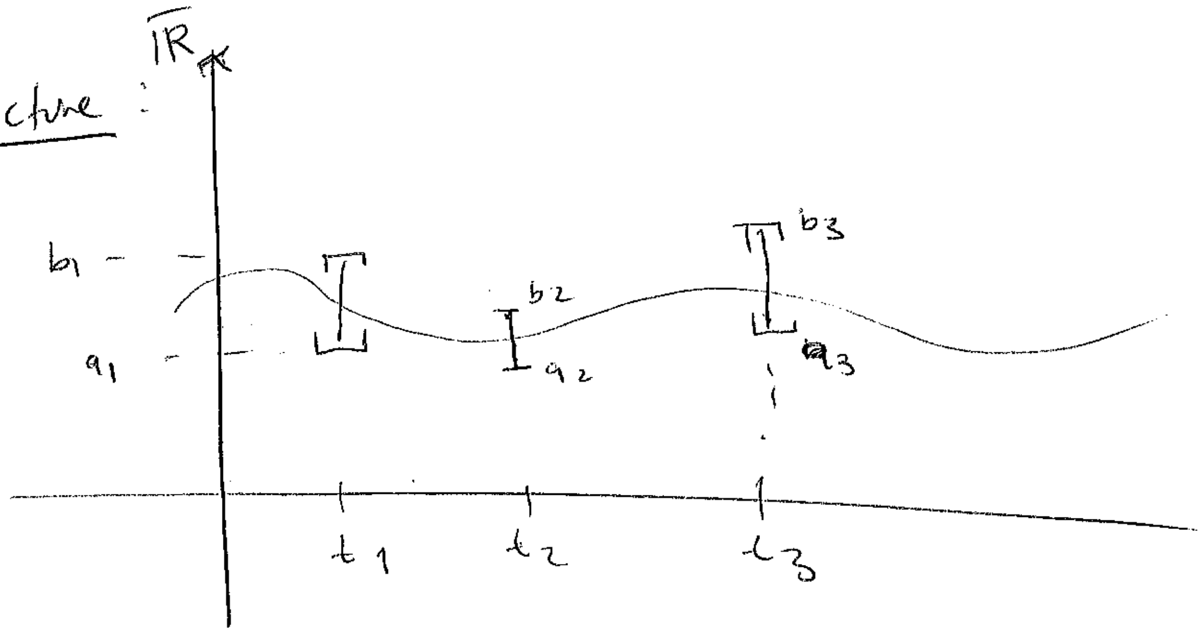
Kolmogorov Extension Theorem: If $\{\mu_{t_1, \dots t_k}; \{t_1, \dots t_k\} \subseteq T\}$ is a consistent system of fidi's then there exists a probability measure P' on $(\overline{\mathbb{R}}^T, \overline{\mathcal{R}}^T)$ so that $\{Z_t, t \in T\}$ is a stochastic process on $(\overline{\mathbb{R}}^T, \overline{\mathcal{R}}^T, P')$ with those fidi's

<u>proof</u>: A "rectangle" in $\overline{\mathbb{R}}^T$ is a set of the form $\{x \in \overline{\mathbb{R}}^T : a_i \le Z_{t_i}(x) < b_i, \ i = 1, \dots k\}$ for $\{t_1, \dots t_k\} \subseteq T$. The finite unions of rectangles form a field, and fidi's specify P' on this field $\overline{\mathcal{R}}_0^T$. We only need to show that P' is countably additive on $\underline{\overline{\mathcal{R}}_0^T}$ (why?)

finite additivity is easy to check.

~~Lemma 8.1 if P is a finitely additive p.m. on a field $\mathcal{F}_0$, then it is countably additive iff $E_n \downarrow \emptyset \Rightarrow P(E_n) \downarrow 0$~~

picture:



(why finitely additive)

Suppose $A_1, \dots A_k$ are all finite dimensional rectangles and they are disjoint.. all these A_i's only depend on finite set of coordinates $\{t_1, \dots t_j\}$

$p'(A_i) \triangleq \mu_{t_1, \dots t_j}(A_i)$

$$p'\left(\bigcup_{i=1}^{k} A_i\right) = \mu_{t_1, \dots t_j}\left(\bigcup_{i=1}^{k} A_i\right)$$

so $\sigma(\overline{\mathcal{R}}_0^T) = \mathcal{R}^T$, Caratheodory extension theorem $\Rightarrow$ there exists a unique extension of p' to $\mathcal{R}^T$ provided we can show that p' is countably additive on $\overline{\mathcal{R}}_0^T$

Lemma 8.1: If P is a finitely additive p.m on a field $\mathcal{F}_0$. then it is countably additive iff

$$E_n \downarrow \phi \Rightarrow P(E_n) \downarrow 0.$$

proof: suppose $A_1, A_2, \dots$ disjoint $\mathcal{F}_0$ sets s.t. $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}_0$.

$$(\Leftarrow) \quad P\left(\bigcup_{n=1}^{\infty} A_n\right) = \left[\lim_{k \to \infty} P\Big(\underbrace{\bigcup_{n=1}^{k} A_n}_{\in \mathcal{F}_0}\Big) + \lim_{k} P\Big(\underbrace{\bigcup_{n=k+1}^{\infty} A_n}_{\in \mathcal{F}_0}\Big)\right] \forall k$$

$$= \lim_{k} \left[\sum_{n=1}^{k} P(A_n)\right] + \underbrace{\lim_{k \to \infty} P\Big(\underbrace{\bigcup_{n=k+1}^{\infty} A_n}_{E_k \downarrow \phi}\Big)}_{= 0 \text{ by hypothesis}}$$

$$= \sum_{n=1}^{\infty} P(A_n)$$

$(\Rightarrow)$ easy.

proof of Kolmogorov extension thm :

Let E_n's be $\bar{\mathcal{R}}_0^T$ set with $E_n \downarrow \phi$
suppose $\lim_{n\to\infty} p'(E_n) = \delta > 0$.

For each n, let D_n be a finite union of closed rectangles, $D_n \subseteq E_n$, with.

$$p(E_n - D_n) < \frac{1}{10^n}\delta$$

Let $C_n = \bigcap_{k=1}^{n} D_k \quad [\subseteq D_n$

Then C_n's are also finite unions of closed rectangles,
$C_n \downarrow \phi$ with $\lim_{n\to\infty} p'(C_n) > \frac{8}{9}\delta$
(since all the stuff we throw away ~~has prob~~
in getting C_n from E_n has prob.
$\leq \delta(1/10 + 1/10^2 + \cdots) = \frac{\delta}{9}$

$E_n = (E_n - D_n) \cup (E_n \cap D_n) = (E_n - C_n) \cup (C_n)$

$p(E_n - C_n) = p\left(\bigcup_{j=1}^{n} (E_n - D_j)\right)$

$p\left(\bigcup_{k=1}^{n}(E_k - D_k)\right) \leq p\left(\bigcup (E_k - D_k)\right) \leq \delta\left(\frac{1}{10^1} + \cdots\right) = \frac{\delta}{10}\cdot\frac{1}{1-\frac{1}{10}} = \frac{\delta}{\cancel{10}}\,\cancel{\frac{10}{9}}$

$p\left(\bigcap_{k=1}^{n}(D_k)\right) \leq p(C_n)$

$$\bigcup_{k=1}^{n} (E_k - D_k) \supseteq \left(E_n \cap \left(\bigcap_{k=1}^{n} D_k\right)^c \right)$$

$$= (E_n) \cap \left(\bigcap D_k \right)^c$$

$$= E_n \cap \left(\bigcup_{k=1}^{n} (D_k)^c \right)$$

$$= \bigcup_{k=1}^{n} \left((E_n \setminus D_k) \right)$$

$$\boxed{E_k \supseteq E_n}$$

Tychonoff finish:

$\mathbb{R}^T$ is a cartesian product of compact sets, hence it is compact, by Tychonoff. Thus any decreasing sequence of non-empty sets C_n which are closed (in the product topology) has non-empty intersection, contradicting $C_n \downarrow \phi$.

Elementary finish: (Diagonal argument)

For each C_n, pick $\underline{x_n} \in C_n$ with $\underline{x_n}$ having only finitely many non-zero coordinates.

All the x_n's are zero except for a (common) countable set of coordinates, $t_1, t_2, t_3 - - - -$

pick a subsequence of $\{x_n\}_{n=1}^{\infty}$ so that the $x_n(\underline{t_1})$ ~~coordinates~~ converge ~~((but leave x...~~ (May be Too

pick a subsubsequence keeping the 1^{st} one of the previous sequence so that $x_n(t_2)$ converge ets.

Get a subsequence st. have convergence at all t_i's

Limit function is in all C_n's so $\bigcap_{n=1}^{\infty} C_n \neq \phi$ $*$

Where do we stand ?

The Kolmogorov extension thm guarantees that there are stochastic processes with the fidi's specified in the examples. But Poisson, Wiener (=Brownian motion) and Gamma processes are also supposed to have some path properties. It turns out that the sets

$$\{x \in \mathbb{R}^T : Z_t(x) \overset{=x(t)}{} \text{ is nondecreasing in } t\}$$

$$\{x \in \mathbb{R}^T : Z_t(x) \overset{=x(t)}{} \text{ is continuous in } t\}$$

$(T = [0,\infty))$

are not even in $\mathcal{R}^T$ (why).

(Also things like "the first time the process hits or exceeds level 7" are not measurable random variables.)

So to get the desired path properties for certain important processes, we need to go back to the drawing board.

$\boxed{\text{Theorem 8.1}}$ Let $\{X_t;\, t\in T\}$ be a stoch. process on $(\Omega, \mathcal{F}, P)$

(i) If $\mathcal{U} \subseteq T$ and $A \in \sigma(X_t,\, t\in \mathcal{U})$, $\omega \in A$

then $X_t(\omega') = X_t(\omega)$ $\forall t\in \mathcal{U} \Rightarrow \omega' \in A$.

(ii) $A \in \sigma(X_t,\, t\in T\} \Rightarrow A \in \sigma(X_t, t\in S)$

for some countable set S.

<u>proof</u>: (i) Let $\mathcal{G}$ = σ-field of subsets of Ω s.t.

$B \in \mathcal{G}$ iff. ω and ω' either <u>both</u> in B

or <u>neither</u> in B

Then all X_t's, $t\in\mathcal{U}$, <u>are</u> $\mathcal{G}$-measurable.

so $\sigma(X_t,\, t\in \mathcal{U}) \subseteq \mathcal{G}$.

(ii) Let $\mathcal{F}_S = \sigma(X_t, t\in S)$ any $S \subseteq T$.

Then $\mathcal{F}_T = \sigma(X_t,\, t\in T) = \bigcup_{\substack{\text{countable} \\ S\subseteq T}} \mathcal{F}_S$ (WFY?)

($\supseteq$ obvious)

Consequence

Let $T = [0,\infty)$ and let $C \subseteq \mathbb{R}^T$ be the set of all continuous functions

then $C \notin \bar{\mathcal{R}}^T$. If it were then theorem 8.1 (ii) implies that there exists a countable $S \subseteq [0,\infty)$ for which $C \in \mathcal{F}_S = \sigma(Z_t : t \in S)$

But if $x \in C$, then theorem 8.1.(i) implies that $y \in C$ for any $y \in \mathbb{R}^T$ with $x(t) = y(t)$. $\forall t \in S$.

But clearly it is possible to get a discontinuous $\underline{y}$ by changing x at a point outside any prespecified countable S. The same reasoning shows that the set of nondecreasing functions is not in $\mathcal{R}^T$.

Construction of Standard Gamma Process.

Construction of Gamma Process

(i) use Kolmogorov exten. theorem to get

$\{\tilde{X}_r, r \in \mathbb{Q}^+\}$ = standard gamma process on time set $\mathbb{Q}^+$. (= non-negative rationals).

(ii) For $0 \le r \le q$, both in $\mathbb{Q}^+$

$\tilde{X}_q - \tilde{X}_r \sim \Gamma(q-r, 1)$ so.

$P(\omega \in \Omega : \tilde{X}(q,\omega) - \tilde{X}(r,\omega) \geq 0\} = 1$

Since $\mathbb{Q}^+ \times \mathbb{Q}^+$ is countable,

$P(\omega \in \Omega : \tilde{X}(r,\omega)$ is non-decreasing in $r \in \mathbb{Q}^+\}$

$= P(\omega \in \Omega : \tilde{X}(q,\omega) \geq \tilde{X}(r,\omega) \quad \forall\ \underline{0 \le r < q} \in \mathbb{Q}^+\} = 1$

(iii) On the ($\mathbb{R}^{\mathbb{Q}^+}$ measurable) set of ω's for which $\tilde{X}(r,\omega)$ not nondecreasing, or for which

$\lim_{r \downarrow 0} \tilde{X}(r,\omega) \neq 0$,

redefine $\tilde{X}(r,\omega) \equiv 0$.

(iv) Set $X(t,\omega) = \lim_{r \downarrow t} \tilde{X}(r,\omega)$, $t \in [0,\infty)$

Then $X(t,\omega)$ is a standard gamma process with nondecreasing (right-continuous) path (all ω) (VFY)

Q: How to get left-continuous paths?

<u>Remark</u>: This same construction works for Poisson processes as well. But one can get a Poisson process ~~⇔~~ iid sequence of exponential r.v's.

Construction of Brownian Motion:

Let D = nonnegative dyadic rationals $(J/2^k)$

~~Define~~ We will define $W(t,\omega) = \lim_{r \downarrow t} W(r,\omega)$, $r \in D$ (after modifying $\tilde{W}$ on a set of Probability 0) and conclude that W is a BM on $[0,\infty)$.

– To get that $\lim_{r \downarrow t} \tilde{W}(r,\omega)$ exists and is continuous in t $\forall \omega$, we need to show that $\tilde{W}(r,\omega)$ is a.s. uniformly continuous in r on bounded intervals of D.

→ Let $I_{n,k} = \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]$

$$B_n = \left\{ \omega : \max_{0 \le k \le n2^n} \sup_{r \in (I_{n,k}) \cap D} \left| \tilde{W}(r,\omega) - \tilde{W}\left(\tfrac{k}{2^n},\omega\right) \right| > \tfrac{1}{n} \right\}$$

If $\omega \notin \underset{\left(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} B_k\right)}{\underline{\limsup B_n}}$, then $\tilde{W}(r,\omega)$ is uniformly continuous on $D \cap [0,t]$ $\forall t$. (Fix $\underline{t} > 0$, and $\varepsilon > 0$, let n be such that $n > t$, $3n^{-1} < \varepsilon$, and $\omega \notin B_n$

then if $r, r' \in D \cap [0,t]$ and $|r' - r| \le \frac{1}{2^n}$ <u>a very easy</u> "3-step" argument shows

$$|\tilde{w}(r,w) - \tilde{w}(r',w)| \le 3n^{-1} < \varepsilon$$

$$\left(\le |\tilde{w}(r,w) - \tilde{w}(\tfrac{k}{2^n},w)| + |\tilde{w}(\tfrac{k}{2^n},w) - \tilde{w}(r',w)| \le \tfrac{2}{n}\right) \ldots$$

which implies uniform continuity on $D \cap [0,t]$ for such w.

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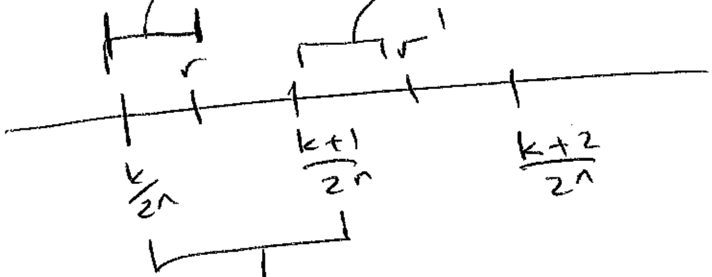
two cases. (i) $r, r' \in$ same $\underline{I_{n,k}}$

$w \notin B_n$.

$$|\tilde{w}(r,w) - \tilde{w}(r',w)| \le |\tilde{w}(r,w) - \tilde{w}(\tfrac{k}{2^n},w)| + |\tilde{w}(r',w) - \tilde{w}(\tfrac{k}{2^n},w)| \le \tfrac{2}{n} < \varepsilon.$$

(ii) $r \in [\frac{k}{2^n}, \frac{k+1}{2^n}]$, $r' \in [\frac{k+1}{2^n}, \frac{k+2}{2^n}]$, then.

$$\begin{aligned} |\tilde{w}(r,w) - \tilde{w}(r',w)| &\le |\tilde{w}(r,w) - \tilde{w}(\tfrac{k}{2^n},w)| \\ &+ |\tilde{w}(\tfrac{k}{2^n},w) - \tilde{w}(\tfrac{k+1}{2^n},w)| \\ &+ |\tilde{w}(r',w) - \tilde{w}(\tfrac{k+1}{2^n},w)| \le \tfrac{3}{n} < \varepsilon. \end{aligned}$$



Lemma 8.2 (Maximal inequality for symmetric Random walk).

Suppose $\boxed{X_1, \ldots X_n \text{ iid.}, \ X_1 \overset{D}{=} -X_1}$

$S_k = \sum_{i=1}^{k} X_i$, then for $\alpha > 0$.

$$P(\max_{k \le n} S_k \geq \alpha) \leq 2P(S_n \geq \alpha)$$

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proof : Obviously

(*) $P(\max_{k \le n} S_k \geq \alpha, S_n \geq \alpha) = P(S_n \geq \alpha)$

Let $A_k = \{\max_{i < k} S_i < \alpha \leq S_k\}$

(= event that r.walk first exceeds α at time k).

$$P(\max_{k \le n} S_k \geq \alpha, \ S_n < \alpha)$$

$$= \sum_{k=1}^{n-1} P(A_k \cap \{S_n < \alpha\})$$

$$\leq \sum_{1}^{n-1} P(A_k \cap \{S_n - S_k < 0\})$$

by indep. of $X_1, \ldots X_k$ and $X_{k+1}, \ldots X_n$ and symmetry

$$= \sum_{k=1}^{n-1} P(A_k \cap \{S_n - S_k > 0\})$$

$$\leq \sum_{k=1}^{n-1} P(A_k \cap \{S_n > \alpha\}) \leq P(S_n \geq \alpha)$$

Add this to (*) above ☒ done

<u>Mills Ratio</u>: if $\phi(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ (N(0,1) density)

$$\Phi(y) = \int_{-\infty}^{y} \phi(t)\,dt \quad (N(0,1) \text{ cdf})$$

then for $y > 0$,

$$\boxed{\left(\frac{1}{y} - \frac{1}{y^3}\right)\phi(y) \leq \underbrace{1 - \Phi(y)}_{P(X>y)} \leq \frac{1}{y}\phi(y) = \frac{1}{y}\cdot\frac{1}{\sqrt{2\pi}} e^{-y^2/2}}$$

<u>proof</u>:

$$e^{-y^2/2}\left(\frac{1}{y} - \frac{1}{y^3}\right) = \int_y^\infty e^{-t^2/2}\left(1 - \frac{3}{t^4}\right)dt$$

$$\leq \underbrace{[1 - \Phi(y)]\sqrt{2\pi} \leq \frac{1}{y}\int_y^\infty t\cdot e^{-t^2/2}\,dt = \frac{1}{y} e^{-y^2/2}}_{\text{important part for most applications.}}$$

Corollary of Maximal inequality:

$$P\left(\max_{k \leq n} |S_k| \geq \alpha\right) \leq 2P(|S_n| \geq \alpha)$$

<u>proof</u> Apply lemma to $S_1, \ldots S_n$ and $\underline{-S_1, \ldots -S_n}$

<u>Lemma</u> $\underline{P(\limsup B_n) = 0}$

enough to show $\sum_{n=1}^{\infty} P(B_n) < \infty$ (Borel-Cantelli 1)

For $\alpha > 0$, $r > 0$, $\delta > 0$ (r, and $r + \delta \in D$)

$$P\left(\max_{\substack{i \\ r \leq i2^{-m} \leq r+\delta}} |\tilde{W}(i2^{-m}) - \tilde{W}(r)| \geq \alpha\right)$$

$$\leq 2P(|\tilde{W}(r+\delta) - \tilde{W}(r)| \geq \alpha) \quad (\text{by } \underline{\text{Corollary above}})$$

[denote $S_i = \tilde{W}(i/2^m) - \tilde{W}(r)$

$$S_{i+1} = \tilde{W}\left(\tfrac{i+1}{2^m}\right) - \tilde{W}(r) = \overbrace{\tilde{W}\left(\tfrac{i+1}{2^m}\right) - \tilde{W}(i/2^n)}^{X_{i+1}} + \overbrace{\tilde{W}(i/2^m) - \tilde{W}(r)}^{S_i}$$

(108)

$\tilde{w}(r+\delta) - \tilde{w}(r) \sim N(0, \underset{\text{var.}}{\delta}) =$

$\leq 4 \cdot P\left(Z > \frac{\alpha}{\sqrt{\delta}} \right) \leq 4 \cdot \frac{\sqrt{\delta}}{\alpha} \cdot \phi\left(\frac{\alpha}{\sqrt{\delta}}\right)$ little ϕ

now let $m \to \infty$;

$$P\left(\sup_{\substack{q \in D \\ r \leq q \leq r+\delta}} |w(q) - w(r)| \geq \alpha \right) \leq 4 \frac{\sqrt{\delta}}{\alpha} \cdot \phi\left(\frac{\alpha}{\sqrt{\delta}}\right)$$

$(\Rightarrow)$ $P(B_n) = P\left(w : \max_{0 \leq k \leq n2^n} \sup_{r \in I_{n,k} \cap D} |\tilde{w}(r,w) - \tilde{w}(\tfrac{k}{2^n}, w)| > \tfrac{1}{n} \right)$

$$\leq \sum_{k=1}^{n2^n} P\left(\sup_{r \in I_{n,k} \cap D} |\tilde{w}(r,w) - \tilde{w}(\tfrac{k}{2^n}, w)| > \tfrac{1}{n} \right)$$

$$\leq \sum_{k=1}^{n2^n} 4 \cdot \frac{\sqrt{2^{-n}}}{n^{-1}} \cdot \phi\left(\frac{1/n}{\sqrt{2^{-n}}} \right)$$

$$= 4 \cdot n^2 2^{n/2} \cdot \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}(2^n/n^2)}$$

$\Rightarrow \sum P(B_n) < \infty$

Finish Construction of BM:

Let $B = (\limsup B_n) \cup (\tilde{w}_0 \neq 0)$

Define $w(t,w) = I_{B^c} \lim_{\substack{r \downarrow t \\ r \in D}} \tilde{w}(r,w)$

VFY: $\{w_t, t \geq 0\}$ is BM.

Definition of Standard BM :

std BM is $\{W_t ; t \geq 0\}$ on (Ω, F, P) s.t.

(i) $P(W_0 = 0) = 1$

(ii) independent increments

(iii) $0 \leq s \leq t \Rightarrow W_t - W_s \sim N(0, Var = (t-s))$

(iv) $\forall \omega \in \Omega$, $W(t, \omega)$ is continuous in t.

New BM's from old. :

proposition : if $\{W_t : t \geq 0\}$ is BM, then

1) $W'(t, \omega) = c^{-1} W(c^2 t, \omega)$ is BM $(c > 0)$

(2) $W''(t, \omega) = \begin{cases} 0 & t = 0 \\ t W(1/t, \omega) & t > 0 \end{cases}$ is BM

proof 1) Properties (i), (ii), (iv) are obvious, For $0 \leq s \leq t$

$$W'(t) - W'(s) = c^{-1} \underbrace{(\underbrace{W(c^2 t, \omega) - W(c^2 s, \omega)}_{N(0,\ c^2(t-s) = var)})}_{N(0,\ var = t-s)}.$$

2). (i) and (iv) are obvious except for continuity at 0.

For $0 \leq s \leq t$

$$E(W(t) W(s)) = E(W^2(s) + \overset{\text{indep.}}{W(s)[W(t) - W(s)]}) = s = s \wedge t.$$

So: $E(W''(s)W''(t)) = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = st\,\frac{1}{t} = s.$

Since the increments $W''(t_i) - W''(t_{i-1})$, $1 \le i \le k$

are jointly Normal with mean zero (since linear transformation of $\{W(1/t_i)$, $1 \le i \le k\}$) (ii) and (iii) follow.

(Note: Joint distribution of Normals is determined by mean, variances, + covariances)

For continuity at 0, apply the maximal inequality for symmetric random walk.

Strong Markov Property:

Direct proof of continuity of $\boxed{t\,W(1/t)}$ (110 b)

By SLLN $\frac{B(n)}{n} \to 0$ as $n \to \infty$ through the integers. To handle the values between integers, use Kolmogorov's inequality

$$P\left(\sup_{0 \le k \le 2^m} \left| B\left(n + \frac{k}{2^m}\right) - B(n) \right| > n^{2/3} \right)$$

$$\le n^{-4/3} \, E\left((B_{n+1} - B_n)^2 \right)$$

letting $m \to \infty$ we get

$$P\left(\sup_{u \in [n, n+1]} |B_u - B_n| > n^{2/3} \right) \le n^{-4/3}.$$

since $\sum_{n=1}^{\infty} n^{-4/3} < \infty$, Borel-Cantelli I implies

that $\frac{B_u}{u} \to 0$ as $u \to \infty$. taking $u = 1/t$. we have

$X_t = t\,B(1/t) \to 0$ as $t \to 0$..