Math 7810-110 Spring 2011 Final Exam

Instructor: Erkan Nane
Due Tuesday May 3, 2011.
Put in my mailbox by 5:00pm.

Problem 1. Use a Martingale convergence to prove the following 0-1 law. Let $\{\mathcal{F}_n\}_{n\geq 1}$ be an increasing sequence of σ-algebras and $\{\mathcal{G}_n\}_{n\geq 1}$ be a deacreasing sequence of σ-algebras with $\mathcal{G}_1 \subset \sigma(\cup_{n=1}^{\infty}\mathcal{F}_n)$. Suppose that $\mathcal{F}_n$ and $\mathcal{G}_n$ are independent for each n. Show that if $A \in \cap_{n=1}^{\infty}\mathcal{G}_n$, then $P(A) = 0$ or 1.

Problem 2. Let $X_1, X_2, \cdots$ be iid random variables, $S_n = \sum_{i=1}^{n} X_i$. We define a strict change of sign as a time at which the process goes from positive values to negative values or from negative values to positive values. Is there any common distribution for the X_i's for which

$$0 < P(S_0, S_1, S_2, \cdots \text{ has finitely many strict sign changes}) < 1$$

Prove your answer.

Problem 3. Let $X_1, X_2, \cdots$ be iid random variables with $E(X_k) = 0$ and $E(X_k^2) = 1$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Is it necessarily true that

$$E\frac{S_n^+}{\sqrt{n}} \to \sqrt{\frac{1}{2\pi}}.$$

Give proof or counterexample.

Problem 4. Let X_t and Y_t $t \geq 0$ be independent, standard Brownian motions.
a. Find $P(X_t > Y_t \text{ for some } t \in (0, 1])$
b. Find $P(X_t > Y_t \text{ for some } t \in [1/2, 1])$

Problems from the book(Durrett 4th edition): 5.7.4, 5.7.5, 5.7.6, 8.1.2, 8.2.1, 8.2.2, 8.2.3, 8.2.4