Math 7810-110 Spring 2011 Exam #1

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Due Friday March 11, 2011.

Problem 1. Let X_1 and X_2 be independent Bernoulli(p) random variables, $0 < p < 1$. (So $P(X_i = 1) = 1 - P(X_i = 0) = p$, $i = 1, 2$). Let $X_3 = 1_{\{X_1 = X_2\}}$. Find

(a) $E(X_3|X_1)$

(b) $E(X_1|X_3)$

(c) $E\big(E(X_3|X_1)|X_1\big)$

(d) $E\big(E(X_3|X_1)|X_2\big)$

(e) $E(X_3|X_1, X_2)$

Problem 2. Let $X_1, X_2, \cdots$ be random variables on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{S}$ be a sub-σ-field of $\mathcal{F}$. If $E(|X_n|) \to 0$ as $n \to \infty$, is it necessarily true that

$$E|E(X_n|\mathcal{S})| \to 0 \ \text{ as } \ n \to \infty.$$

Problem 3. Let $X_1, X_2, \cdots$ be iid random variables with $E(X_k) = 0$ and $E(X_k^2) = 1$. Let $S_n = X_1 + X_2 + \cdots + X_n$. Prove that

$$E\left|\frac{S_n}{\sqrt{n}}\right| \to \sqrt{\frac{2}{\pi}}.$$

Problem 4. If $X_n \to X$ a.s., $E(X_n) \to c < \infty$, and $P(X_n > 1) = 1$ for all n then $E(\ln(X_n)) \to E(\ln(X))$.

Problem 5. Let $X_1, X_2, \cdots$ be independent non-negative random variables each of mean 1. Define $M_0 = 1$, and for $n \in \mathbb{N}$ let

$$M_n = X_1 X_2 \cdots X_n.$$

Then M_n is a martingale, so that

$$M_\infty = \lim_{n\to\infty} M_n \ \text{ exists } a.s.$$

Show that the following five statements are equivalent:

(i) $E(M_\infty) = 1$

(ii) $M_n \to M_\infty$ in L^1

(iii) M_n is uniformly integrable

(iv) $\prod_{n=1}^{\infty} a_n > 0$ where $0 < a_n := E(X_n^{\frac{1}{2}}) \leq 1$

(v) $\sum_{n=1}^{\infty}(1 - a_n) < \infty$

If one (then everyone) of the above five statements fails to hold, then

$$P(M_\infty = 0) = 1.$$

Problem 6.(a) Let $X_1, X_2, \cdots$ be iid, $P(X_i = 0) = P(X_i = 1) = \frac{1}{2}$. Let $Y_n = \prod_{i=1}^{n} X_i$.

Are the random variables $Y_n, n \geq 1$, uniformly integrable?

(b) Same question, but with $X_i's$ iid $Uniform[0, 2]$

(c) Same question, but with $X_i's$ iid $Uniform[0, 3/2]$

Problems from the book(Durrett 4th edition): 5.4.8, 5.4.9, 5.4.10, 5.5.1, 5.5.3, 5.5.5, 5.5.7.