Math 7800-110 Fall 2010 Final Exam

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Due Wednesday December 8, 2010.

Problem 1. Suppose that $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ be sequences of random variables such that

$$X_n \implies X \text{ and } Y_n \implies Y \text{ as } n \to \infty.$$

Suppose that X_n and Y_n are independent for all n and that X and Y are independent. Then show that

$$X_n + Y_n \implies X + Y \text{ as } n \to \infty.$$

Problem 2. Let $X \geq 0$ be a random variable. Assume that $\sum_{n=1}^{\infty} P(X > a_n) < \infty$ where $\{a_n\}_{n\geq 0}$ denote a sequence of numbers such that $a_0 = 0$, $a_{n+1} > a_n$ and $\frac{a_n}{n} \uparrow \infty$. Let $Y_n = X \cdot 1_{\{X<a_n\}}$, $n \geq 1$. Prove the following

a. $\sum_{m=1}^{\infty} mP(a_{m-1} \leq X < a_m) < \infty$.

b. For every $N < n$ we have

$$\frac{\sum_{m=1}^{n} E(Y_m)}{a_n} \leq \frac{n \cdot E(Y_N)}{a_n} + \sum_{m=N+1}^{n} \frac{m}{a_m} \cdot E(X \cdot 1_{\{a_{m-1}\leq X<a_m\}})$$

Hint: Observe that $\sum_{m=1}^{n} E(Y_m) \leq n \cdot E(Y_n)$. Also use $\frac{n}{a_n} \leq \frac{m}{a_m}$ if $m \leq n$.

c. $\dfrac{\sum_{m=1}^{n} E(Y_m)}{a_n} \to 0$ as $n \to \infty$.

Problem 3. Suppose that $X_1, X_2, \cdots$ is a sequence of IID random variables and assume that $E(X_1) = 0$ and $E(X_1^2) = 1$.

a. Prove that $\frac{X_n}{\sqrt{n}} \to 0$ as $n \to \infty$, a.s. Then show that in fact

$$\frac{\max_{1\leq k\leq n}\{|X_k|\}}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty, \ a.s.$$

Hint for the second part: Show that for any sequence of numbers $\{a_n\}_{n\geq 1}$:

$$\frac{a_n}{\sqrt{n}} \to 0, \text{as } n \to \infty \text{ implies } \frac{\max_{1\leq k\leq n}\{|a_k|\}}{\sqrt{n}} \to 0, \quad \text{as } n \to \infty.$$

b. Let $X_{k,n} = X_k 1_{\{|X_k|<\sqrt{n}/2\}}$, $1 \leq k \leq n$. Prove that

(i) $\sum_{k=1}^{n} \frac{X_{k,n}}{\sqrt{n}} - \sum_{k=1}^{n} \frac{X_{k,n}^2}{2\cdot n}$ converges in distribution to $Y \sim N(-\frac{1}{2}, 1)$. Hint: You may use the CLT for $\{X_n\}_{n\geq 1}$ and part a.

(ii)

$$\sum_{k=1}^{n} \frac{|X_{k,n}|^3}{n^{3/2}} \to 0 \text{ as } n \to \infty, a.s.$$

c. Prove that $\prod_{k=1}^{n}(1 + \frac{X_k}{\sqrt{n}})$ converges in distribution to e^Y where $Y \sim N(-\frac{1}{2}, 1)$. Hint: Uset the following inequality (follows from Taylor's expansion)

$$|\log(1 + y) - y + \frac{y^2}{2}| \leq |y|^3, \ |y| < 1/2$$

Observe that we may have $\frac{X_k}{\sqrt{n}} < -1$.

Problem 4. Let X be a non-degenerate random variable (i.e., it is not a constant). Define $\varphi_X(\lambda) = E(\lambda^2 X^2 \wedge 1)$, $\lambda \in R$. Here $a \wedge b$ is the minimum of the numbers $a, b \in R$.
a. Prove : (i) $\varphi_X(\lambda)$ is continuous
(ii) $E(X^2) = \lim_{\lambda \to 0} \frac{\varphi_X(\lambda)}{\lambda^2} = \sup_{|\lambda|>0} \frac{\varphi_X(\lambda)}{\lambda^2}$
b. Prove: If $\sum_{k=1}^{\infty} \varphi_X(a_k) < \infty$ then $\sum_{k=1}^{\infty} a_k^2 < \infty$.
Hint: Prove first that $a_k \to 0$ and then use part a.
c. Prove: If $E(X^2) < \infty$ and $\sum_{k=1}^{\infty} a_k^2 < \infty$ then $\sum_{k=1}^{\infty} \varphi_X(a_k) < \infty$.
d. Suppose $X_1, X_2, \cdots$ are independent, symmetric random variable (i.e. X_k has the same distribution as $-X_k$). Prove: $\sum_{k=1}^{\infty} a_k X_k$ converges a.s. if and only if $\sum_{k=1}^{\infty} \varphi_{X_k}(a_k) < \infty$.

Problem 5. Let $X_k \sim Uniform[-k, k]$ for $k = 1, 2, \cdots$ and assume that $X_1, X_2, \cdots$ are independent. Does

$$\frac{\sum_{k=1}^{n} X_k}{\sqrt{\sum_{k=1}^{n} var(X_k)}}$$

converge in probability? Does it converge in distribution? Justify your answers.

Problem 6. Let $X_1, X_2, \cdots$ be iid $N(0,1)$, $S_n = \sum_{k=1}^{n} X_k$
a. Find $P(\sum_{n=1}^{\infty} 1_{\{|X_n| < \frac{1}{n}\}} < \infty)$.
b. Find $P(\sum_{n=1}^{\infty} 1_{\{|S_n| < \frac{1}{n}\}} < \infty)$.

Problem 7. Suppose that $X_n \implies X$, where random variable X has a density with respect to Lebesgue measure. Let $< t >$ be the fractional part of the number t, for example, $< \pi > = < 3.141592 \cdots > = 0.141592 \cdots$ Does it follow that $< X_n > \implies < X >$?

Problem 8. Let X be a random variable with $P(X = 1) = P(X = -1) = 1/2$. Show that if Y is another random variable independent of X, then $X + Y$ cannot be $N(0, \sigma^2)$ for any variance $\sigma^2 \geq 0$.

Problem 9. Let $X_1, X_2, \cdots$ be symmetric (meaning $+X_k$ and $-X_k$ have the same distribution, for all k), not identically distributed, independent random variables. Suppose $S_n^{(1)} = X_1 + \cdots + X_n$ converges to a finite limit as $n \to \infty$. Show that

$$S^{(m)} = X_1^m + \cdots + X_n^m$$

converges a.s. to a finite limit for $m = 2, 3, \cdots$ i.e. for any integer $m \geq 2$.

Problem 10. Suppose that $\varphi(t) = f(t) + ig(t)$ is a characteristic function, with real part $f(t)$ and imaginary part $g(t)$.
a. Is $f(t)$ necessarily a characteristic function? Proof or counterexample.
b. Is $|\varphi(t)|^2 = f^2(t) + g^2(t)$ necessarily a characteristic function? Proof or counterexample.