Math 7800-110 Fall 2010 Exam #1

Instructor: Erkan Nane

Due Friday October 8, 2010.

Problem 1. Let f be a nonnegative measurable function on $(X, \mathcal{F}, \mu)$. Prove that for any $0 < p < \infty$,

$$\int_X f(x)^p d\mu(x) = p \int_0^\infty \lambda^{p-1} \mu\{x \in X : \ f(x) > \lambda\} d\lambda.$$

Problem 2. Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose f and g are two nonnegative functions satisfying the following inequality; There exists a constant C such that for all $\epsilon > 0$ and $\lambda > 0$,

$$\mu\{x \in X : \ f(x) \geq 2\lambda, \ g(x) \leq \epsilon\lambda\} \leq C\epsilon^2 \mu\{x \in X : \ f(x) > \lambda\}.$$

Prove that

$$\int_X f(x)^p d\mu(x) \leq C_p \int_X g(x)^p d\mu(x)$$

for any $0 < p < \infty$ for which both integrals are finite where C_p is a constant depending on C and p.

Problem 3. Let $(X, \mathcal{F}, \mu)$ be a measure space and let $1 < p < \infty$. Assume that $f : X \to \mathbb{R}$ is measurable and satisfies

$$\mu\{x \in X : \ |f(x)| > y\} \leq \frac{c_0}{y^p}, \quad c_0 \ \text{ independent of } y > 0.$$

Let $1 \leq r < p$. Show that

$$\int_X |f|^r d\mu \leq c\mu(X)^{1-r/p},$$

where c depends only on c_0, r, p.

(Hint; The left side$=r \int_0^\infty \cdots dy = r \int_0^R + r \int_R^\infty .$)

Problem 4. Let f_n be a nonnegative sequence of functions in $L^1[0,1]$ with the property that $\int_0^1 f_n(t)dt = 1$ and $\int_{1/n}^1 f_n(t)dt \leq 1/n$ for all $n = 1, 2, \cdots$. Define $h(x) = \sup_n f_n(x)$. Prove that $h \notin L^1[0,1]$.

Problem 5. Suppose $1 \leq p < \infty$, $f_n, f \in L^p[0,1]$, $f_n \geq 0$ a.e., $\lim_{n\to\infty} f_n = f$ a.e., and $\lim_{n\to\infty} \int_0^1 f_n^p(x)dx = \int_0^1 f^p(x)dx$. Prove that $f_n \to f$ in $L^p[0,1]$.

Problem 6. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $X_n : \Omega \to \mathbb{R}$ be arbitrary random variables. Show that there exists positive constants c_n such that $c_n X_n \to 0$ a.e.

Problem 7. Let $(\Omega, \mathcal{F}, P)$ be a probability space.
a. Suppose $F : \mathbb{R} \to \mathbb{R}$ is a continuous function and $f_n \to f$ in measure. prove that $F(f_n) \to F(f)$ in measure.
b. If $f_n \geq 0$ and $f_n \to f$ in measure. Then

$$\int_\Omega f d\mu \leq \liminf_{n\to\infty} \int_\Omega f_n d\mu.$$

c. Suppose $|f_n| \leq g$ where $g \in L^1(P)$ and $f_n \to f$ in measure. Then

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu.$$

Problem 8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space and let $f_1, f_2, \cdots, f_n$ be measurable functions. Suppose $1 < p < \infty$. Prove that

$$\int_\Omega |\frac{1}{n}\sum_{j=1}^n f_j(x)|^p d\mu(x) \leq \frac{1}{n}\int_\Omega \sum_{j=1}^n |f_j(x)|^p d\mu(x)$$

and

$$\int_\Omega |\frac{1}{n}\sum_{j=1}^n f_j(x)|^p d\mu(x) \leq \left(\frac{1}{n}\sum_{j=1}^n ||f_j||_p\right)^p.$$

The last inequality is known as generalized Minkowski's inequality.

Problem 9. Let $f \in L^1[0,1]$, and let $F(x) = \int_0^x f(t)dt$. If E is a measurable subset of $[0,1]$, show that
a. $F(E) = \{y : \exists x \in E \text{ with } y = F(x)\}$, image of E under F, is measurable.
b. $m(F(E)) \leq \int_E |f(t)|dt$. Here m denotes the Lebesgue measure on $[0,1]$.

Problem 10.(Monotone Class Theorem) Let $\mathcal{H}$ be a class of bounded functions from a set Ω to $\mathbb{R}$ satisfying the following conditions;
(i) $\mathcal{H}$ is a vector space over $\mathbb{R}$;
(ii) The constant function 1 is an element of $\mathcal{H}$;
(iii) If f_n is a sequence of nonnegative functions in $\mathcal{H}$ such that $f_n \nearrow f$ where f is bounded on Ω, then $f \in \mathcal{H}$.

Then if $\mathcal{H}$ contains the indicator function of every set in a semi-algebra $\mathcal{S}$, then $\mathcal{H}$ contains every bounded $\sigma(\mathcal{S})$-measurable function on Ω.

Problem 11. Use Problem 13 to prove Fubini's Theorem for nonnegative functions. This is also known as Fubuni-Tonelli Theorem; Let $\mu \times \nu$ be a product measure such that
(i) μ and ν are σ-finite measures on $\mathcal{A}$ and $\mathcal{B}$ respectively.
(ii) $f : X \times Y \to \mathbb{R}$ is in $\sigma(\mathcal{A} \times \mathcal{B})$, and f is nonnegative.

Then, if one of the iterated integrals

$$\int_X \left[\int_Y f d\nu\right] d\mu \quad \text{or} \quad \int_Y \left[\int_X f d\mu\right] d\nu \text{ exists}$$

then the function f if $\mu \times \nu$ integrable and

$$\int_{X\times Y} f d(\mu \times \nu) = \int_X \left[\int_Y f d\nu\right] d\mu = \int_Y \left[\int_X f d\mu\right] d\nu$$