Math 5/6680 Spring 2009 Final Exam

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Name:
PID:

Instructions

Read each problem carefully. Show all your work. Credit will only be awarded if your work is included.

Problem	max-points	Points
1	25	
2	30	
3	30	
4	30	
5	20	
6	30	
7	40	
8	25	
bonus problem	20	
total	230+20	

(25pts)**Problem 1.** Assume that (X, Y) is uniformly distributed on

$$D = \{(x, y) : \quad 0 \leq x \leq 1,\ 0 \leq x + y \leq 1\}.$$

(5pts)**a.** Find the joint density function of (X, Y).

(5pts)**b.** Find the marginal density function of X.

(5pts)**c.** Find the density function of $Z = X + Y$.

(5pts)**d.** Find $P[Y \leq 2X]$.

(5pts)**e.** Find $E(Y|X)$.

(30pts)**Problem 2.** Let X be uniformly distributed over $(0,1)$ and Y has the density function

$$f_Y(y) = e^{-y}, \text{ if } y \geq 0. \quad f_Y(y) = 0, \text{ if } y < 0.$$

(10pts)**a.** What are the moment generating functions of X and Y.

(10pts)**b.**Assume that X and Y are independent. Find the density function of $X+Y$.

(10pts)**c.** Assume that X and Y are independent. Find the moment generating function of $2X + Y$.

(30pts)**Problem 3.** Let X be a random variable with the density function

$$f(x) = \begin{cases} 1/x^2, & \text{if } x \geq 1 \\ 0, & \text{otherwise.} \end{cases}$$

(5pts)**a.** What is the cumulative distribution of X?

(5pts)**b.** What is the median of X?

(10pts)**c.** Let $Y = \frac{1}{1+X^2}$. What is the density of Y?

(10pts)**d.** How could a random variable Y be generated from a uniform random variable?

(30pts)**Problem 4.** Let X and Y be independent, standard normal random variables. Recall that their common density is the function

$$f(a) = \frac{1}{\sqrt{2\pi}} e^{-a^2/2}, \quad -\infty < a < \infty$$

(10pts)**a.** Compute the (joint) density function of the random vector (X, Y).

(20pts)**b.** Compute the density of the random variable Z, where

$$Z = (X^2 + Y^2)^{1/5}.$$

(Hint use polar coordinates with $dxdy = rdrd\theta$, and $r^2 = x^2 + y^2$).

(20pts)**Problem 5. a.**(5pts) Show that sum of independent normal random variables is a normal random variable using moment generating functions. (Hint: If $X \sim N(\mu, \sigma^2)$, then $M_X(t) = \exp(t\mu + t^2\sigma^2/2)$)

(15pts)**b.** You have 25 light bulbs whose lifetimes are independent normal random variables with mean 100 (hours) and standard deviation 5 (hours). Find the probability that your 25 light bulbs together live for at least 2502 hours.

(30pts)**Problem 6.** Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with the density function

$$f(x) = \begin{cases} 2x, \ 0 \leq x \leq 1, \\ 0, \ \text{oterwise} \end{cases}$$

(15pts)**a.** Approximate

$$P\left[\frac{X_1 + X_2 + \cdots + X_{100}}{100} < 0.40\right].$$

(Write your answer up to numbers that can be obtained from Tables).

(15pts)(**b.** Let $X_1, \cdots, X_n$ be i.i.d. random variables with density function f.
What is

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} X_i, \text{in probability?}$$

(40pts)**Problem 7.** Consider the birth or disaster model with transition probabilities $p(i, i+1) = p_i$ for $i = 0, 1, 2$, $p(i, 0) = q_i$ for $i = 0, 1, 2, \cdots$, here $0 < p_i, q_i < 1$ and $p_i + q_i = 1$.
(5pts)**a.** What is the transition matrix?or give the diagram of transitions.

(10pts)**b.** Find the invariant probability vector in terms of one of the coordinates.

(5pts)**c.** Show that an invariant probability vector exists if and only if

$$C = 1 + p_0 + p_0p_1 + p_0p_1p_2 + \cdots < \infty.$$

(5pts)**f.** What is the invariant probability vector?

(5pts)**d.** When is the chain positive recurrent?

(5pts)**e.** If $p_i = p$ is a constant what is the invariant vector?

(5pts)**g.** Suppose the chain starts at 0. What is the expected first return time to state 0 if $p_i = p$ a constant?

(25pts)**Problem 8.** Consider a Markov chain with state space $\{1, 2, 3, 4, 5\}$ and transition matrix

$$\begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/5 & 4/5 \\ 0 & 0 & 0 & 2/5 & 3/5 \\ 1 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \end{pmatrix}$$

(5pts)**a.** Is this chain irreducible? Is it aperiodic?

(10pts)**b.** Find the stationary probability vector.

(5pts)**c.** Suppose $X_0 = 1$. What is the expected number of steps until the chain is state 1 again?

(5pts)**d.** Suppose $X_0 = 1$. What is the expected number of steps until the chain is in state 4? You can give your answer in terms of matrices without taking inverses(if only this takes long).

(20pts)**Bonus Problem.** Let X_n be a Markov chain on a finite state space S with transition matrix P, and consider the stochastic process Y_n defined by $Y_k = X_{2k}$. Show that Y_n is a Markov chain. (Hint: you need to show that the Markov property holds)