A Brief Review

Consider a linear, homogeneous, constant coefficient second order equation

\[ \frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = 0. \]

The characteristic polynomial corresponding to this equation is

\[ r^2 + ar + b = 0 \]

whose roots are \( r_1 \) and \( r_2 \) where

\[ r_1 = \frac{-a + \sqrt{a^2 - 4b}}{2} \quad r_2 = \frac{-a - \sqrt{a^2 - 4b}}{2} . \]

We will consider three distinct cases (depending on the value of \( a^2 - 4b \)):

I. Two distinct real roots.
If \( 0 < a^2 - 4b \) there are two real distinct roots \( r_1 \neq r_2 \) and the homogenous solution is

\[ x_h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}. \]

II. A double root.
If \( a^2 - 4b = 0 \) there are two equal roots \( r_1 = r_2 \) and the homogenous solution is

\[ x_h(t) = c_1 e^{rt} + c_2 t e^{rt}. \]

III. Complex conjugate roots.
If \( a^2 - 4b < 0 \) there are two complex conjugate roots

\[ r_1 = \alpha + i\beta \quad r_2 = \alpha - i\beta \]

where \( \alpha = -\frac{a}{2} \), \( \beta = \frac{\sqrt{4b - a^2}}{2} \) and \( i \) is the square root of -1. The homogenous solution is

\[ x_h(t) = c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t). \]

Examples:

1. Consider the differential equation \( \frac{d^2 x}{dt^2} - \frac{dx}{dt} - 6 = 0 \). The characteristic polynomial is
\[ r^2 - r - 6 = 0. \] The roots of the characteristic polynomial are
\[
> \text{solve}(r^2-r-6=0);
\]
\[ 3, -2 \]
so the solution of the equation is \( x_h(t) = c_1 e^{3t} + c_2 e^{-2t}. \)

2. Consider the differential equation \( \frac{d^2 x}{dt^2} + 6 \frac{dx}{dt} + 9 = 0. \) The characteristic polynomial is
\[ r^2 + 6r + 9 = 0. \] The roots of the characteristic polynomial are
\[
> \text{solve}(r^2+6*r+9=0);
\]
\[ -3, -3 \]
so the solution of the equation is \( x_h(t) = c_1 e^{-3t} + c_2 t e^{-3t}. \)

3. Consider the differential equation \( \frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 5 x = 0. \) The characteristic polynomial is
\[ r^2 + 4r + 5 = 0. \] The roots of the characteristic polynomial are
\[
> \text{solve}(r^2+4*r+5=0);
\]
\[ -2 \pm i, -2 - i \]
Note, Maple writes \( i \) for \( \sqrt{-1} \), so the solution of the equation is
\[ x_h(t) = e^{-2t} \left( c_1 \cos(t) + c_2 \sin(t) \right). \]

4. Consider the initial value problem \( \frac{d^2 x}{dt^2} + 4 x = 0, x(0) = 1, \frac{dx}{dt}(0)=1. \) The characteristic polynomial is \( r^2 + 4 = 0. \) The roots of the characteristic polynomial are
\[
> \text{solve}(r^2+4=0);
\]
\[ 2i, -2i \]
so the general solution of the equation is \( x_h(t) = c_1 \cos(2t) + c_2 \sin(2t). \) We now use the initial conditions to solve for \( c_1 \) and \( c_2. \)
\[
> \text{sol}:=c[1]*\cos(2*t)+c[2]*\sin(2*t);
\]
\[
> \text{sol} := c_1 \cos(2t) + c_2 \sin(2t)
\]
\[
> \text{eq1}:=\text{subs}(t=0,\text{sol})=1;
\]
\[
> \text{eq1} := c_1 \cos(0) + c_2 \sin(0) = 1
\]
\[
> \text{eq2}:=\text{subs}(t=0,\text{diff(sol,t)})=1;
\]
\[
> \text{eq2} := -2 c_1 \sin(0) + 2 c_2 \cos(0) = 1
\]
\[
> \text{solve}([\text{eq1},\text{eq2}],[c[1],c[2]]);
\]
\[
> \text{solve}([\text{eq1},\text{eq2}],[c[1],c[2]]) = \left\{ c_1 = 1, c_2 = \frac{1}{2} \right\}
\]
so the solution is \( x(t) = \cos(2t) + \frac{\sin(2t)}{2}. \)

\textbf{Undetermined Coefficients}
This method applies to special classes of nonhomogeneous second order equations. It is **crucial** that the homogeneous problem have constant coefficients.

Consider a nonhomogeneous constant coefficient second order equation

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + bx = f(t).
\]

If the right hand side \(f(t)\) has the form (exponential times a polynomial times a trigonometric polynomial) we guess a particular solution of the same form

\[
f(t) = e^{kt}(a_n l^n + a_{n-1} l^{n-1} + \ldots + a_0)(\cos(\omega t) + \sin(\omega t)).
\]

Then guess a particular solution of the form

\[
x_p(t) = e^{kt}(A_n l^n + A_{n-1} l^{n-1} + \ldots + A_0)\cos(\omega t) + e^{kt}(B_n l^n + B_{n-1} l^{n-1} + \ldots + B_0)\sin(\omega t).
\]

If the above solution \(x_p\) is a solution of the homogeneous equation you need to multiply it by \(t^s\) (\(s\) counts the number of times \(x_p\) is a solution of the homogeneous problem, and for a second order equation \(s\) is either 1 or 2).

**Example:**

Consider the differential equation

\[
\frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 5 x = \cos(t).
\]

The characteristic polynomial is 

\[
r^2 + 4r + 5 = 0.
\]

The roots of the characteristic polynomial are

\[
-2 + i, -2 - i
\]

so the solution of the homogeneous problem is 

\[
x_h(t) = e^{-2t}(c_1 \cos(t) + c_2 \sin(t))
\]

we now have to find one particular solution of

\[
\frac{d^2 x}{dt^2} + 4 \frac{dx}{dt} + 5 x = \cos(t)
\]

in order to obtain the general solution.

The idea is to find this solution using an intelligent guess. The second derivative of a cosine term is again a cosine term, but the first one is a sine term. Therefore we try an expression 

\[
x_p(t) = A \cos(t) + B \sin(t)
\]

as initial guess. The goal is to find the undetermined coefficients \(A\), \(B\) (in \(x_p(t)\)) in such a manner that \(x_p\) is the desired particular solution.

```maple
> restart;
> guess := y(t) = A*cos(t)+B*sin(t);
guess := y(t) = A \cos(t) + B \sin(t)

> param := {A, B};
param := \{A, B\}

> eq := diff(y(t), t$2) + 4*diff(y(t), t) + 5*y(t) = cos(t);
eq := \frac{d^2 y(t)}{dt^2} + 4 \left( \frac{d}{dt} y(t) \right) + 5 y(t) = \cos(t)

> subs(guess, eq);
\frac{d^2}{dt^2} (A \cos(t) + B \sin(t)) + 4 \left( \frac{d}{dt} (A \cos(t) + B \sin(t)) \right) + 5 A \cos(t) + 5 B \sin(t)
```
\[ = \cos(t) \]

\[ > \text{simplify}(%); \quad 4A \cos(t) + 4B \sin(t) - 4A \sin(t) + 4B \cos(t) = \cos(t) \]

Since \( \cos(t) \) and \( \sin(t) \) are linearly independent, the coefficient \( 4A + 4B \) of the cosine on the left hand side of the equation has to be equal to 1 (the coefficient of the cosine on the right), whereas the coefficient \( 4B - 4A \) of the sine has to be zero. This leads to two linear equations which we solve by

\[ > \text{sloveparam}:=\text{solve}\{4A+4B=1,4B-4A=0\},\text{param}; \]

\[ \text{sloveparam}:= \left\{ B = \frac{1}{8}, A = \frac{1}{8} \right\} \]

Thus, \( x_p(t) = \frac{\cos(t) + \sin(t)}{8} \) is a particular solution. We can check this by using the \text{odetest}-command.

\[ > \text{odetest}(y(t)=(\cos(t)+\sin(t))/8,D(D(y))(t)+4*D(y)(t)+5*y(t)=\cos(t)); \]

\[ 0 \]

Therefore the general solution is given as \( x_{\text{gen}}(t) = x_h(t) + x_p(t) \), i.e.

\[ x_{\text{gen}}(t) = e^{-2t} \left( c_1 \cos(t) + c_2 \sin(t) \right) + \frac{\cos(t) + \sin(t)}{8}, \text{a fact, which you can also test by inserting this expression into the d.e.:} \]

\[ > \text{odetest}(y(t)=e^{-2*t}*(c[1]*\cos(t)+c[2]*\sin(t))+(\cos(t)+\sin(t))/8,D(D(y))(t)+4*D(y)(t)+5*y(t)=\cos(t)); \]

\[ 0 \]

If in addition we were given initial conditions, we would now use those to solve for the constants \( c_1 \) and \( c_2 \).

\section*{Exercise}

Find the general solution of \( \frac{d^2x}{dt^2} - 3 \frac{dx}{dt} - 4x = 2 \sin(t) \).

\[ > \text{restart}; \]

\section*{Exercise}

Find the general solution of \( \frac{d^2x}{dt^2} + x(t) = 2 \sin(t) \).

\[ > \text{restart}; \]

\section*{A Trick}

What went wrong? Clearly, \( A \cos(t) + B \sin(t) \) solves the homogeneous problem.

\[ > \text{odetest}(y(t)=A*\cos(t)+B*\sin(t),D(D(y))(t)+y(t)=0); \]

\[ 0 \]

Try the guess \( x_p(t) = t \left( A \cos(t) + B \sin(t) \right) \) and see what you get:

Let us explore this last guess somewhat more. Consider the nonhomogeneous problem
\[
\frac{d^2 x}{dt^2} + p \frac{dx}{dt} + q x(t) = a \cos(\beta t) + b \sin(\beta t)
\]

where \(p\) and \(q\) are real numbers, \(\beta > 0\) and \(0 < a^2 + b^2\). We ask for conditions under which the guess 
\[
y(t) := t (A \cos(\beta t) + B \sin(\beta t))
\]
works. To this end we insert \(y\) into the differential equation.

```
> restart;
> y(t):=t*(A*cos(beta*t)+B*sin(beta*t));
```

\[
y(t) := t (A \cos(\beta t) + B \sin(\beta t))
\]

```
> diff(y(t),t$2)+p*diff(y(t),t)+q*y(t)=a*cos(beta*t)+b*sin(beta*t);
```

This yields the following four conditions for \(p\), \(q\), \(A\) and \(B\):

\[
\begin{align*}
  t \cos(\beta t) & : -A \beta^2 + p B \beta + q A = 0, \\
  t \sin(\beta t) & : -B \beta^2 - p A \beta + q B = 0, \\
  \cos(\beta t) & : 2 B \beta + p A = a, \\
  \sin(\beta t) & : -2 A \beta + p B = b.
\end{align*}
\]

> solve({-A*beta^2+p*B*beta+q*A=0, -B*beta^2-p*A*beta+q*B=0, 2*B*beta+p*A=a, -2*A*beta+p*B=b}, {q,p,B,A});

\[
\begin{align*}
  p &= 0, \\
  B &= \frac{1}{2} \frac{a}{\beta}, \\
  q &= \frac{b}{\beta}, \\
  A &= -\frac{1}{2} \frac{b}{\beta}.
\end{align*}
\]

Note that \(a \cos(\beta t) + b \sin(\beta t)\) solves \(\frac{d^2 x}{dt^2} + \beta^2 x(t) = 0\) as the following shows

```
> with(DEtools):
> odetest(z(t)=a*cos(beta*t)+b*sin(beta*t),D(D(z))(t)+beta^2*z(t)=0);
```

\[
0
\]

consequently, one concludes that the guess \(t (A \cos(\beta t) + B \sin(\beta t))\) only works in case that \(A \cos(\beta t) + B \sin(\beta t)\) solves the homogeneous problem.

### The Resonance Case
Consider a second order linear differential equation

\[
\frac{d^2 x}{dt^2} + a \frac{dx}{dt} + b x = f(t)
\]

where \(a\), \(b\) are real numbers, and \(f(t)\) has the form \(f(t) = p_n(t) e^{\alpha t} \cos(\beta t) + q_n(t) e^{\alpha t} \sin(\beta t)\) with \(\alpha\), \(\beta\) real numbers and \(p_n\), \(q_n\) polynomials in \(t\) of degree less than or equal to \(n\).

**Step 1.** Set up standard trial function.

**Step 2.** Check whether any term solves the homogeneous d.e.
Step 3. If so, multiply by $t$ and go to Step 2.
Step 4. If not, determine coefficients by inserting the improved trial function and its derivatives into the de.

**Guesses for Other Forcing Functions**
Find a particular solution for the following differential equations.

\begin{itemize}
  \item[a)] \[
  \frac{d^2 x}{dt^2} - \frac{dx}{dt} - 6 x(t) = 10 e^{2t}
  \]
  
  \[
  > \texttt{restart;}
  \]

  \item[b)] \[
  \frac{d^2 x}{dt^2} - 3 \frac{dx}{dt} = 64 (t^3 - t^2)
  \]
  
  \[
  > \texttt{restart;}
  \]

  \item[c)] \[
  \frac{d^2 x}{dt^2} + 4 x(t) = 2 t^2 \sin(2t)
  \]
  
  \[
  > \texttt{restart;}
  \]
\end{itemize}