

Math 2650

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Exact Differential Equations

Consider the differential equation

$$M(t, x) + N(t, x) \frac{dx}{dt} = 0$$

or (equivalently) written in differential form the equation

$$M(t, x) dt + N(t, x) dx = 0.$$

Theorem. Let the functions M, N, M_x, N_t (where subscripts denote partial derivatives) be continuous in a simply connected region R (of the $t - x$ plane). The above differential equation is an exact differential equation in R if and only if

$$M_x(t, x) = N_t(t, x)$$

at each point of R . That is, there exists a function Φ satisfying

$$\Phi_t(t, x) = M(t, x) \quad \text{and} \quad \Phi_x = N(t, x)$$

if and only if M and N satisfy $M_x(t, x) = N_t(t, x)$.

If the equation

$$M(t, x) dt + N(t, x) dx = 0$$

is exact then an explicit solution is given by

$$\Phi(t, x) = c$$

where Φ satisfies $\Phi_t(t, x) = M(t, x)$ and $\Phi_x = N(t, x)$.

This implicit solution may be constructed as follows:

$$\Phi(t, x) = \int M(t, x) dt + h(x)$$

and

$$\begin{aligned} \Phi_x(t, x) &= \frac{\partial}{\partial x} \left(\int M(t, x) dt \right) + \frac{d}{dx} h(x) \\ &= \int M_x(t, x) dt + \frac{d}{dx} h(x) \end{aligned}$$

which yields

$$\frac{d}{dx} h(x) = N(t, x) - \left(\int M_x(t, x) dt \right)$$

hence

$$\Phi(t, x) = \int M(t, x) dt + \int \left(N(t, x) - \left(\int M_x(t, x) dt \right) \right) dx.$$

Example

1. Consider

$$(x \cos(t) + 2 t e^x) + (\sin(t) + t^2 e^x - 1) \frac{dx}{dt} = 0$$

we first check that this is in fact an exact differential equation

```
> diff(x*cos(t)+2*t*exp(x), x);  
cos(t) + 2 t e^x  
> diff(sin(t)+t^2*exp(x)-1, t);  
cos(t) + 2 t e^x
```

so it is exact. Now lets calculate the solution

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> int(x*cos(t)+2*t*exp(x), t);  
x sin(t) + t^2 e^x
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so

$$\Phi(t, x) = x \sin(t) + t^2 e^x + h(x)$$

and

$$\frac{d}{dx} h(x) = \sin(t) + t^2 e^x - 1 - (\sin(t) + t^2 e^x)$$

so

$$\frac{d}{dx} h(x) = -1$$

and

$$h(x) = -x.$$

Therefore an implicit solution is given by

$$x \sin(t) + t^2 e^x - x = c.$$

Problems

Determine whether each of the equations is exact, if it is find the solution.

1. $(2x + 3) + (2y - 2) \frac{dy}{dx} = 0$

2. $(2x + 4y) + (2x - 2y) \frac{dy}{dx} = 0$

3. $(3t^2 - 2tx + 2) dt + (6x^2 - t^2 + 3) dx = 0$

4. $(2tx^2 + 2x) dt + (2t^2x + 2t) dx = 0$