Math 2650

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Exact Differential Equations

Consider the differential equation

$$M(t,x) + N(t,x)\frac{dx}{dt} = 0$$

or (equivalently) written in differential form the equation

$$M(t,x)\,dt + N(t,x)\,dx = 0.$$

Theorem. Let the functions M, N, M_x, N_t (where subscripts denote partial derivatives) be continuous in a simply connected region R (of the $t - x$ plane). The above differential equation is an exact differential equation in R if and only if

$$M_x(t,x) = N_t(t,x)$$

at each point of R. That is, there exists a function Φ satisfying

$$\Phi_t(t,x) = M(t,x) \qquad \text{and} \qquad \Phi_x = N(t,x)$$

if and only if M and N satisfy $M_x(t,x) = N_t(t,x)$.

If the equation

$$M(t,x)\,dt + N(t,x)\,dx = 0$$

is exact then an explicit solution is given by

$$\Phi(t,x) = c$$

where Φ satisfies $\Phi_t(t,x) = M(t,x)$ and $\Phi_x = N(t,x)$.

This implicit solution may be constructed as follows:

$$\Phi(t,x) = \int M(t,x)\,\mathrm{d}t + h(x)$$

and

$$\Phi_x(t,x) = \frac{\partial}{\partial x}\left(\int M(t,x)\,\mathrm{d}t\right) + \frac{d}{dx}h(x)$$
$$= \int M_x(t,x)\,\mathrm{d}t + \frac{d}{dx}h(x)$$

which yields

$$\frac{d}{dx}h(x) = N(t,x) - \left(\int M_x(t,x)\,\mathrm{d}t\right)$$

hence

$$\Phi(t,x) = \int M(t,x)\,\mathrm{d}t + \int\left(N(t,x) - \left(\int M_x(t,x)\,\mathrm{d}t\right)\right)\mathrm{d}x.$$

Example

1. Consider

$$(x\cos(t) + 2\,t\,e^{x}) + (\sin(t) + t^{2}\,e^{x} - 1)\frac{dx}{dt} = 0$$

we first check that this is in fact an exact differential equation

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> diff(x*cos(t)+2*t*exp(x),x);
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$$\cos(t) + 2\,t\,e^{x}$$

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> diff(sin(t)+t^2*exp(x)-1,t);
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$$\cos(t) + 2\,t\,e^{x}$$

so it is exact. Now lets calculate the solution

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> int(x*cos(t)+2*t*exp(x),t);
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$$x\sin(t) + t^{2}\,e^{x}$$

so

$$\Phi(t, x) = x\sin(t) + t^{2}\,e^{x} + h(x)$$

and

$$\frac{d}{dx}h(x) = \sin(t) + t^{2}\,e^{x} - 1 - \left(\sin(t) + t^{2}\,e^{x}\right)$$

so

$$\frac{d}{dx}h(x) = -1$$

and

h(x)=-x.

Therefore an implicit solution is given by

$$x\sin(t) + t^{2}\,e^{x} - x = c.$$

Problems

Determine whether each of the equations is exact, if it is find the solution.

1. $(2\,x + 3) + (2\,y - 2)\frac{dy}{dx} = 0$

2. $(2\,x + 4\,y) + (2\,x - 2\,y)\frac{dy}{dx} = 0$

3. $\left(3\,t^{2} - 2\,t\,x + 2\right)\,dt + \left(6\,x^{2} - t^{2} + 3\right)\,dx = 0$

4. $\left(2\,t\,x^{2} + 2\,x\right)\,dt + \left(2\,t^{2}\,x + 2\,t\right)\,dx = 0$