# Math 2650

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## Periodic functions

```
> restart:with(plots):
Warning, the name changecoords has been redefined
> f1(t):=sin(2*t);
```

$$f1(t) := \sin(2\,t)$$

```
> f2(t):=cos(3*t);
```

$$f2(t) := \cos(3\,t)$$

```
> f3(t):=cos(5*t+2);
```

$$f3(t) := \cos(5\,t+2)$$

```
> f4(t):=sin(2^(1/2)*t);
```

$$f4(t) := \sin\left(\sqrt{2}\,t\right)$$

```
> f5(t):=cos(Pi*t);
```

$$f5(t) := \cos(\pi\,t)$$

```
> f1(t);f2(t);f3(t);f4(t);f5(t);
```

$$\sin(2\,t)$$

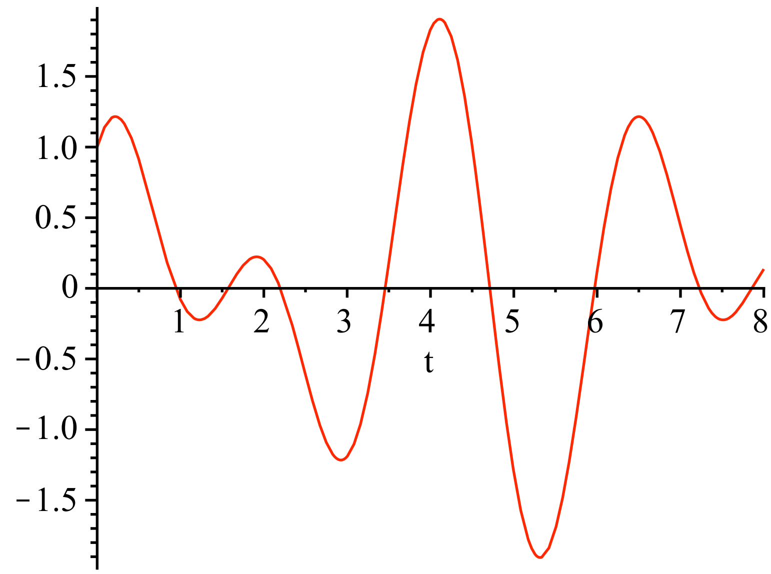
$$\cos(3\,t)$$

$$\cos(5\,t+2)$$

$$\sin\left(\sqrt{2}\,t\right)$$

$$\cos(\pi\,t)$$

```
> plot(f1(t)+f2(t),t=0..8);
```




## Phase amplitude form

In order to better see the behavior of solutions of second order equations we can use the phase amplitude form of of periodic functions. That is given a periodic function of the form

$$f(t) = a\cos(\omega t) + b\sin(\omega t)$$

we can write it as

$$f(t) = A\cos(\omega t - \delta)$$

where

$$A = \sqrt{a^2 + b^2}$$

and

$$\delta = \arctan\left(\frac{b}{a}\right)$$

(note this last equation has two solutions which differ by $\pi$, you must chose the correct solution).

Also recall the two trigonometric identities

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

and

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$$

using these identities we can often rewrite periodic functions to better see the phenomenon of beats.

For example, consider the equation $D^{(2)}(x)(t) + 9\,x(t) = \sin(2\,t)$ with initial conditions *?*

```
> restart:with(DEtools):
> sol:=rhs(dsolve({D((D)(x))(t)+9*x(t) = sin(2*t),x(0)=1,D(x)
  (0)=0},x(t)));
```

$$sol := -\frac{2}{15}\sin(3\,t) + \cos(3\,t) + \frac{1}{5}\sin(2\,t)$$

Using the above identities we have that the solution can be written as

$$\frac{1 \sin(2t)}{5} + \frac{\sqrt{229}\cos(3t + 0.13255)}{15}$$

since

```
> sqrt((2/15)^2+1);evalf(arctan(2/15));
```

$$\frac{1}{15}\sqrt{229}$$

$$0.1325515323$$

## Free oscillation

**undampped oscillations**

Consider the differential equation $\frac{d^2 x}{dt^2} + 16x = 0$. With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0)=1$.

The characteristic equation is $r^2 + 16 = 0$. This equation has the solutions:

```
> restart:
> solve(r^2+16=0);
```

$$4\,I, -4\,I$$

As you know, the homogeneous problem has the general solution $x_h = c_1 \cos(4t) + c_2 \sin(4t)$.

We now find $c_1$ and $c_2$

```
> c1c2:= {c1,c2};
```

$$c1c2 := \{c1, c2\}$$

```
> sol:=c1*cos(4*t)+c2*sin(4*t);
```

$$sol := c1\cos(4t) + c2\sin(4t)$$

```
> eq1:=subs(t=0,sol);
```

$$eq1 := c1\cos(0) + c2\sin(0)$$

```
> eq2:=subs(t=0,diff(sol,t));
```

$$eq2 := -4\,c1\sin(0) + 4\,c2\cos(0)$$

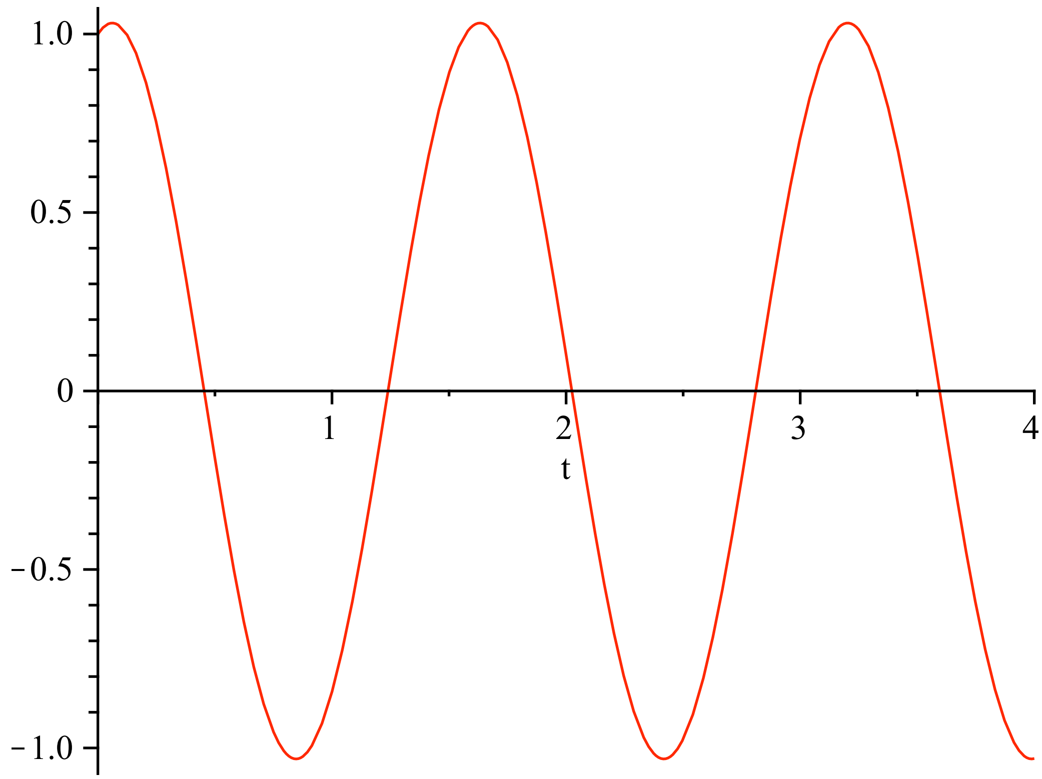
```
> solve({eq1=1,eq2=1},c1c2);
```

$$\left\{c2 = \frac{1}{4}, c1 = 1\right\}$$

```
> sol:=cos(4*t)+(1/4)*sin(4*t);
```

$$sol := \cos(4t) + \frac{1}{4}\sin(4t)$$

```
> plot(sol,t=0..4);
```




**underdampped oscillations**

Consider the differential equation $\frac{d^2 x}{dt^2} + 2\frac{dx}{dt} + 16x = 0$. With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0)=1$.

The characteristic equation is $r^2 + 2r + 16 = 0$. This equation has the solutions:

```
> restart:
> solve(r^2+2*r+16=0);
```

$$-1 + I\sqrt{15},\ -1 - I\sqrt{15}$$

As you know, the homogeneous problem has the general solution
$x_h = c_1 e^{-t} \cos(\sqrt{15}\, t) + c_2 e^{-t} \sin(\sqrt{15}\, t)$.
We now find $c_1$ and $c_2$

```
> c1c2:= {c1,c2};
```

$$c1c2 := \{c1, c2\}$$

```
> sol:=c1*exp(-t)*cos(sqrt(15)*t)+c2*exp(-t)*sin(sqrt(15)*t);
```

$$sol := c1\, e^{-t} \cos(\sqrt{15}\, t) + c2\, e^{-t} \sin(\sqrt{15}\, t)$$

```
> eq1:=subs(t=0,sol);
```

$$eq1 := c1\, e^{0} \cos(0) + c2\, e^{0} \sin(0)$$

```
> eq2:=subs(t=0,diff(sol,t));
```

$$eq2 := -c1\, e^{0} \cos(0) - c1\, e^{0} \sin(0)\sqrt{15} - c2\, e^{0} \sin(0) + c2\, e^{0} \cos(0)\sqrt{15}$$

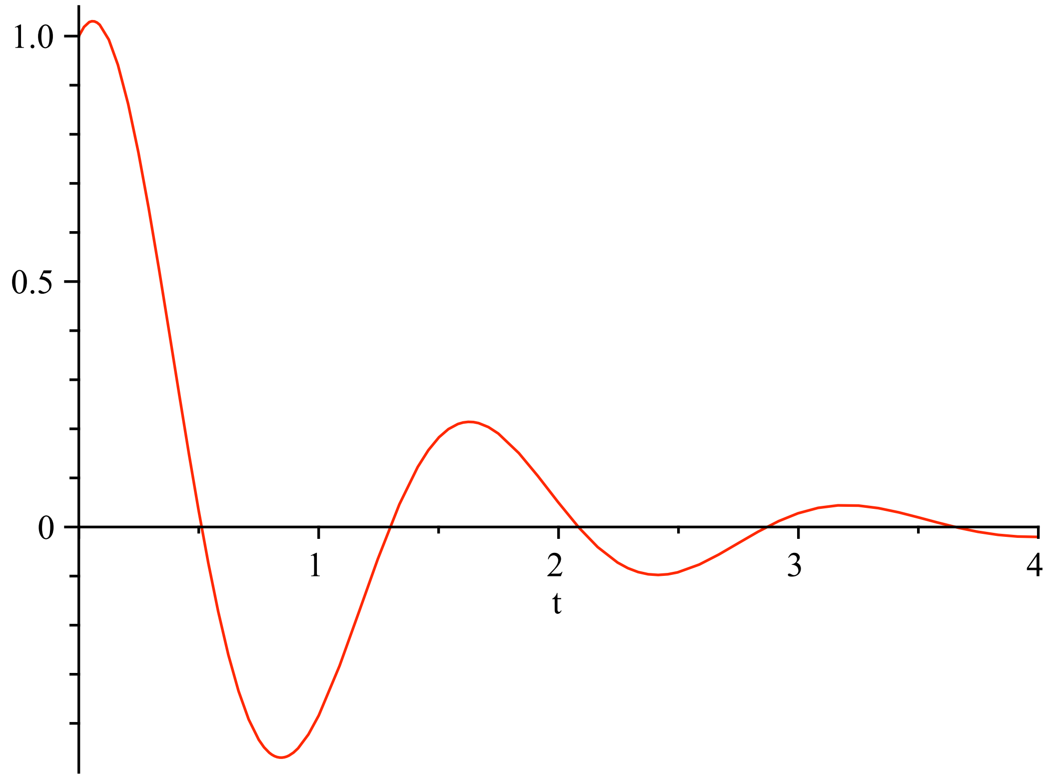
```
> solve({eq1=1,eq2=1},c1c2);
```

$$\left\{c1 = 1, c2 = \frac{2}{15}\sqrt{15}\right\}$$

```
> sol:=exp(-t)*(cos(4*t)+(2/sqrt(15))*sin(4*t));
```

$$sol := e^{-t}\left(\cos(4\,t) + \frac{2}{15}\sqrt{15}\,\sin(4\,t)\right)$$

```
> plot(sol,t=0..4);
```




**critically dampped oscillations**

Consider the differential equation $\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16\,x = 0$. With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0)=1$.

The characteristic equation is $r^2 + 8\,r + 16 = 0$. This equation has the solutions:

```
> restart:
> solve(r^2+8*r+16=0);
```

$$-4, -4$$

As you know, the homogeneous problem has the general solution $x_h = c_1\,e^{-4\,t} + c_2\,t\,e^{-4\,t}$.

We now find $c_1$ and $c_2$

```
> c1c2:= {c1,c2};
```

$$c1c2 := \{c2, c1\}$$

```
> sol:=c1*exp(-4*t)+c2*t*exp(-4*t);
```

$$sol := c1\,e^{-4\,t} + c2\,t\,e^{-4\,t}$$

```
> eq1:=subs(t=0,sol);
```

$$eq1 := c1\, e^{0}$$

```
> eq2:=subs(t=0,diff(sol,t));
```

$$eq2 := -4\, c1\, e^{0} + c2\, e^{0}$$

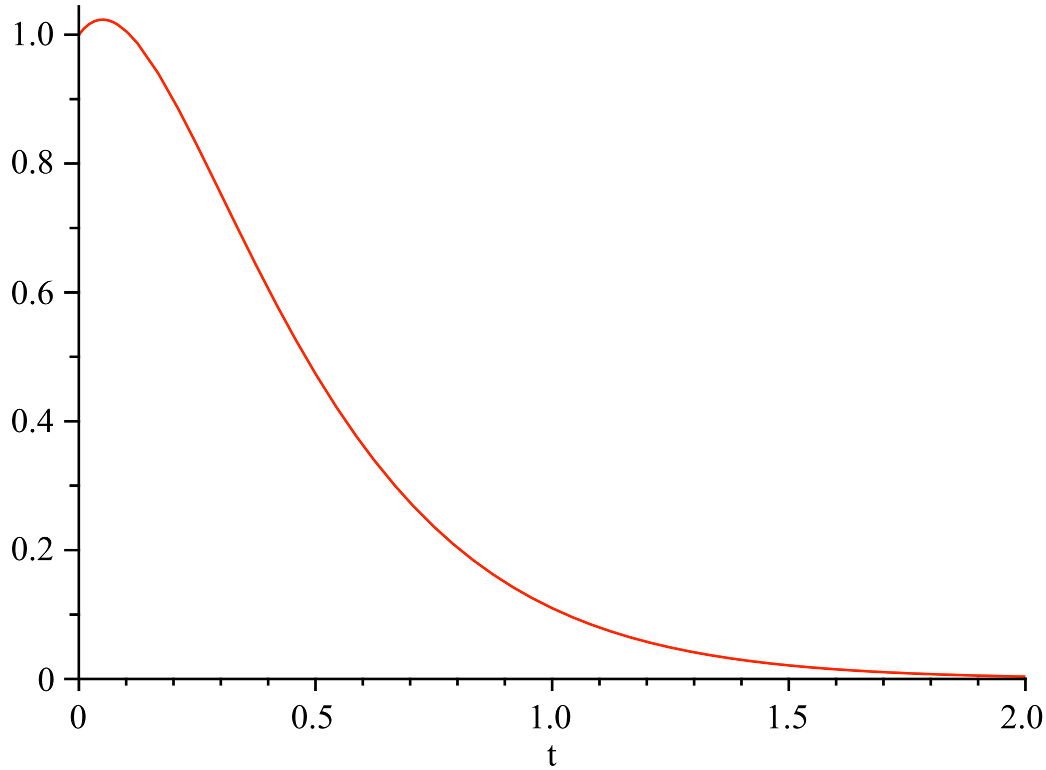
```
> solve({eq1=1,eq2=1},c1c2);
```

$$\{c1 = 1, c2 = 5\}$$

```
> sol:=exp(-4*t)+5*t*exp(-4*t);
```

$$sol := e^{-4t} + 5\, t\, e^{-4t}$$

```
> plot(sol,t=0..2);
```




**overdampped oscillations**

Consider the differential equation $\frac{d^2 x}{dt^2} + 10\,\frac{dx}{dt} + 16\,x = 0$. With initial conditions $x(0) = 1$ and $\frac{dx}{dt}(0)=1$.

The characteristic equation is $r^2 + 10\,r + 16 = 0$. This equation has the solutions:

```
> restart:
> solve(r^2+10*r+16=0);
```

$$-2, -8$$

As you know, the homogeneous problem has the general solution $x_h = c_1\, e^{-2t} + c_2\, e^{-8t}$.

We now find $c_1$ and $c_2$

```
> c1c2:= {c1,c2};
```

$$c1c2 := \{c1, c2\}$$

```
> sol:=c1*exp(-2*t)+c2*exp(-8*t);
```

$$sol := c1\,e^{-2t} + c2\,e^{-8t}$$

```
> eq1:=subs(t=0,sol);
```

$$eq1 := c1\,e^{0} + c2\,e^{0}$$

```
> eq2:=subs(t=0,diff(sol,t));
```

$$eq2 := -2\,c1\,e^{0} - 8\,c2\,e^{0}$$

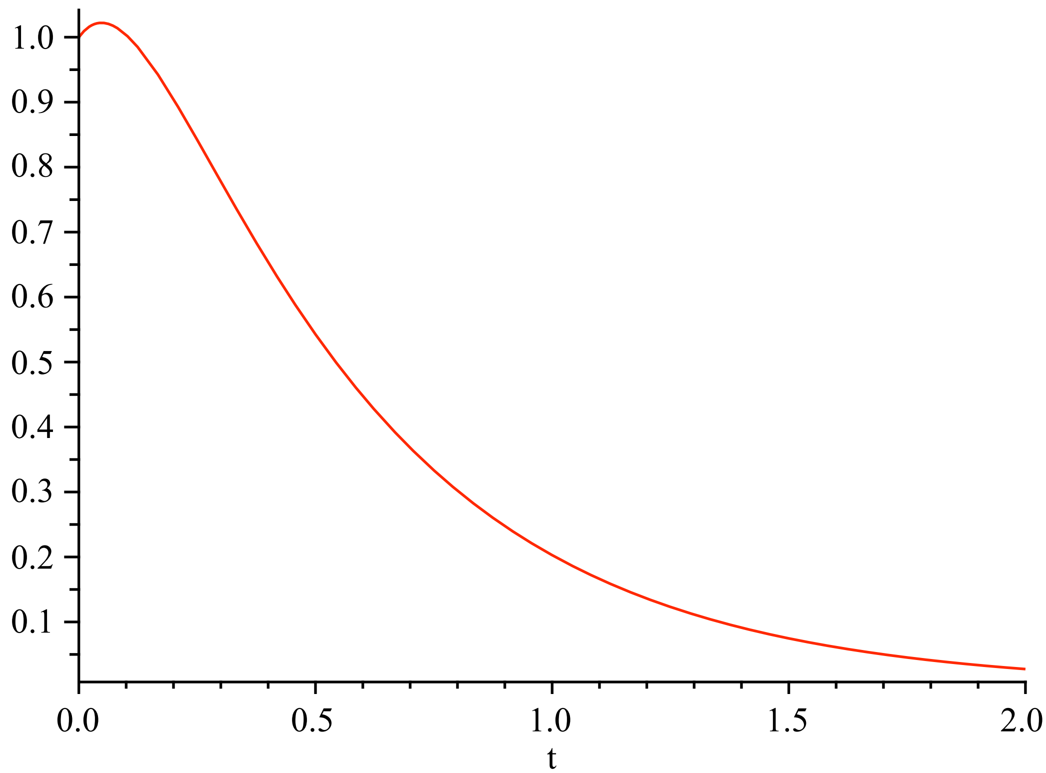
```
> solve({eq1=1,eq2=1},c1c2);
```

$$\left\{c2 = -\frac{1}{2},\ c1 = \frac{3}{2}\right\}$$

```
> sol:=(3/2)*exp(-2*t)-(1/2)*exp(-8*t);
```

$$sol := \frac{3}{2}\,e^{-2t} - \frac{1}{2}\,e^{-8t}$$

```
> plot(sol,t=0..2);
```




```
>
```